Isolating the singularities of the plane projection of a generic space curve

George Krait, Sylvain Lazard, Guillaume Moroz, Marc Pouget

Isolating the singularities of the plane projection of a generic space curve

George Krait[1], Sylvain Lazard[1], Guillaume Moroz[1], and Marc Pouget[1]

Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy

Abstract

Isolating the singularities of a plane curve is the first step towards computing its topology. For this, numerical methods are efficient but not certified in general. We are interested in developing certified numerical algorithms for isolating the singularities. In order to do so, we restrict our attention to the special case of plane curves that are projections of smooth curves in higher dimensions. This type of curves appears naturally in robotics applications and scientific visualization. In this setting, we show that the singularities can be encoded by a regular square system whose solutions can be isolated with certified numerical methods. Our analysis is conditioned by assumptions that we prove to be generic using transversality theory, we also provide a semi-algorithm to check their validity.

Keywords: Transversality, Generic Singularities, Certified Numerical Algorithms, Interval Analysis, Singular Curve Topology

1. Introduction

The problem of computing the topology of a real plane curve consists in computing a piecewise-linear graph that can be deformed continuously toward that curve. Such a problem is critical for drawing plane curves with the correct topology. A natural approach to compute the topology of a singular curve is first to isolate its singular points, second to compute the topology in a neighbourhood of those points and third to compute the topology of the smooth remaining part of that curve. One of the main challenges for this goal is to isolate the singular points efficiently and correctly. The aim of this paper is to do so with certified numerical methods and we show that this could be achieved for the specific class of plane curves that are projections of C^∞ smooth curves in higher dimension.

By certified algorithm, we refer to algorithms that always output mathematically correct results in a given model of computation; for instance, randomized Las-Vegas algorithms are (usually) certified, but randomized Monte-Carlo algorithms are not; numerical methods that may miss solutions or output spurious solutions are not certified. We consider in this paper the RAM model of computation. Recall that the singular points of a plane curve, defined by the equation $f(x, y) = 0$, are the solutions of the system defined by $f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$; it should be stressed that this system is over-determined, i.e., it has more equations than variables, which prevents us to use certified numerical methods such as interval Newton methods [MKC09a]. To the best of our knowledge, no efficient and certified algorithm is known for isolating the singularities of any plane curve.

Email address: 1 (Firstname.Name@inria.fr)

Main contributions. In this paper, we present a square and regular system that encodes the singularities of the plane projection of a C^∞ smooth curve in $\mathbb{R}^n$, which may not be algebraic (Theorems 36 & 51). This system can thus be solved with state-of-the-art certified numerical methods based on interval arithmetic or certified homotopy tracking. However it encodes the singularities of the plane projection only if some assumptions are satisfied (defined in Section 2.2). Our second main result is the proof that those assumptions are satisfied generically using transversality theory (Section 3). Finally, we present Semi-algorithm 3 that checks whether a given curve satisfies our assumptions, i.e. an algorithm that stops if and only if the assumptions are satisfied. The combination of these results provides a method that is both efficient and certified for computing the singularities of the plane projection of a generic curve.

We also address the case of curves that are the silhouettes of smooth surfaces in $\mathbb{R}^n$ (Definition 26). Such curves naturally appear in parametric systems since they partition the parametric space with respect to the number of solutions of the system. For such curves, all our results directly hold except their genericity for which we were only able to prove some partial results (Section 3.3). Our contribution is a generalization of [IMP16b] that only considers the 3-dimensional case and is in the same spirit as the work of Delanoue et al. [DL14].

State of the art. The problem of isolating the singularities of a plane curve is a special case of the problem of isolating the solutions of a zero-dimensional system in $\mathbb{R}^2$. We give below a concise state of the art of certified methods for these two problems, organized in two main classes.

Symbolic methods. Symbolic methods are widely used for solving in a certified way zero-dimensional systems. Classical such methods are based on Gröbner bases, resultant theory and univariate representations (see e.g., [CLO92, BPR06]). In this context, methods dedicated to the bivariate case have also been designed (see [Hon96, GVK96, BLM$^+$16, vdHL18] and references therein). The drawback of such methods is, however, that they are not efficient compared to numerical methods and that they do not handle non-algebraic curves.

Certified numerical methods. When a zero-dimensional system is regular (Definition 50), its solutions can be isolated in a certified way using interval-arithmetic subdivision methods [Neu91, MKC09b] or homotopy approaches with certified path tracking (see [BL13] and references therein). However, these methods do not directly work for isolating the singularities of plane curves because the system $f(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ that encodes the singularities of the curve $f(x,y) = 0$ is neither square nor regular. To the best of our knowledge only two contributions present certified numerical approaches for isolating the singularities of plane curves: Delanoue and Lagrange [DL14] consider the apparent contour of a smooth mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ and Imbach et al. [IMP16b] handle plane projections of smooth curves in $\mathbb{R}^3$.

The rest of the paper is organized as follows: In Section 2, we introduce notation and the assumptions we consider in our approach. In Section 3, we prove the genericity of our assumptions, with a focus on the case of silhouette curves in Section 3.3. In Section 4, we introduce the Ball system that characterizes the singularities of the plane projection and we prove that it is regular at its solutions. Finally, in Section 5, we provide a semi-algorithm to check the assumptions introduced in Section 2.

2. Notation and Assumptions

2.1. Preliminaries

For a positive integer n, a closed (resp. an open) n-box is the Cartesian product of n closed (resp. open) interval. Assume that $n \geqslant 3$ and let B be an open n-box and $\overline{B}$ be the topological closure of B with respect to the usual topology in $\mathbb{R}^n$. Let $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ denote the set of smooth functions (i.e., differentiable infinitely many times) from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. Consider the function $P = (P_1, \ldots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. We will denote by $\mathfrak{C}$ (resp. $\overline{\mathfrak{C}}$) the solution set of the system $\{P_1(x) = \cdots = P_{n-1}(x) = 0\}$, with $x = (x_1, \ldots, x_n) \in B$ (resp. with $x \in \overline{B}$). Also, consider the projection $\pi_{\mathfrak{C}}$ (resp. $\pi_{\overline{\mathfrak{C}}}$) from $\mathfrak{C}$ (resp. $\overline{\mathfrak{C}}$) to the (x_1, x_2)-plane. Unless otherwise stated, the plane projection of a point $x \in \mathbb{R}^n$ is (x_1, x_2). If $\overline{\mathfrak{C}}$ is a smooth curve (see the definition below), define $\mathfrak{L}_{\mathrm{c}}$ (resp. $\mathfrak{L}'_{\mathrm{c}}$) to be the set of points q in $\mathfrak{C}$ (resp. $\overline{\mathfrak{C}}$) such that the tangent line, denoted by $T_q\mathfrak{C}$, (resp. $T_q\overline{\mathfrak{C}}$) is orthogonal to the (x_1, x_2)-plane. We also define the set $\mathfrak{L}_{\mathrm{n}}$ (resp. $\mathfrak{L}'_{\mathrm{n}}$) to be the set of points q in $\mathfrak{C}$ (resp. $\overline{\mathfrak{C}}$) such that the cardinality of the pre-image of $\pi_{\mathfrak{C}}(q)$ under $\pi_{\mathfrak{C}}$ (resp. $\pi_{\overline{\mathfrak{C}}}$) is at least two. We will see later that, under some generic assumption, $\mathfrak{L}_{\mathrm{c}}$ (resp. $\mathfrak{L}_{\mathrm{n}}$) is equal to the set of points in $\mathfrak{C}$ that project to a cusp (resp. node) which justifies the subscript c (resp. n).

Regular and singular points [Dem00, Definition 2.2.2]. Let $m \geqslant 1$ be an integer, V be a subset of $\mathbb{R}^m$ and $p \in V$. We call p a regular (or smooth) point of V if V is a sub-manifold at p, that is, there exist a neighbourhood W of p in $\mathbb{R}^m$, an integer $k > 0$ and k smooth functions $\varphi_1, \ldots, \varphi_k$ defined over W, such that $V \cap W$ is the set of solutions of $\{\varphi_1(x) = \cdots = \varphi_k(x) = 0\}$ in W and the rank of the matrix $\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \cdots & \frac{\partial \varphi_k}{\partial x_m} \end{pmatrix}$, evaluated at q, is k. We call this matrix the Jacobian matrix of the system $\{\varphi_1(x) = \cdots = \varphi_k(x) = 0\}$ and we denote it by $J_{(\varphi_1, \ldots, \varphi_k)}$. If q is not a regular point of V, we call it a singular point. If all points in V are regular, then V is called regular or smooth. Otherwise, V is called singular.

For $\varphi = (\varphi_1, \ldots, \varphi_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$, we denote by $T_q\varphi$ its derivative (also known as the tangent map) at the point q. Note that the Jacobian matrix $J_\varphi = J_{(\varphi_1, \ldots, \varphi_k)}$ is the expression of the derivative in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^k$.

Definition 1. *Let f be a real smooth function at $a \in \mathbb{R}$. The order of f at a is the integer* $\mathrm{ord}_a(f(x)) = \min\{k \in \mathbb{N} \mid \frac{\partial^k f}{\partial x^k}(a) \neq 0\}$ *if it exists, otherwise* $\mathrm{ord}_a(f(x)) = \infty$. *For the case $a = 0$, we write for simplicity* $\mathrm{ord}(f) = \mathrm{ord}_a(f)$.

Multiplicity in zero-dimensional systems.

Definition 2 ([CLO05, Definition 4.2.1])**.** *For integers $m \geqslant n \geqslant 1$, let $G = (g_1(x), \ldots, g_m(x))$ be a polynomial function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and q be a solution of the system $\{G = 0\}$. Let $\mathbb{R}[x]$ be the ring of polynomials with n variables and define $\mathbb{R}[x]_q = \{\frac{h_1}{h_2} \mid h_1, h_2 \in \mathbb{R}[x], h_2(q) \neq 0\}$ the localization of $\mathbb{R}[x]$ at q. Define the intersection multiplicity of q in the system $\{G = 0\}$ (or equivalently the multiplicity of the system $\{G = 0\}$ at q) to be the dimension of the real vector space $\frac{\mathbb{R}[x]_q}{I_G}$, where I_G is the ideal generated by the set $\{\frac{g_1}{1}, \ldots, \frac{g_m}{1}\}$ in $\mathbb{R}[x]_q$.*

The previous definition is classical for the algebraic case. However, in our paper, we are interested in curves defined as the zero locus of smooth functions. For this goal, we consider a more general definition for a system $S = \{f_1(x) = \cdots = f_m(x) = 0\}$ with $f_i \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Let a be a solution of S and k be a non-negative integer, we define the dual space of rank k, denoted by $D_a^k[S]$, to be the vector space of all linear combinations c of differential functionals $\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}$ with $k_1 + \cdots + k_n \leqslant k$ such that:

(a) $D_a^0[S] = span(\{\frac{\partial^0}{\partial x_1^0 \ldots \partial x_n^0}\})$,

(b) c applied to f_i, evaluated at a is zero for all integers $1 \leqslant i \leqslant m$, and

(c) for all $i \in \{1, \ldots, n\}$, the anti-differentiation transformation ϕ_j applied to c is in $D_a^{k-1}[S]$. The anti-differentiation transformation ϕ_j is the linear operator mapping the order h differential functional $\frac{\partial^h}{\partial x_1^{h_1}\ldots\partial x_j^{h_j}\ldots\partial x_n^{h_n}}$ to the order $(h-1)$ differential functional $\frac{\partial^{h-1}}{\partial x_1^{h_1}\ldots\partial x_j^{h_j-1}\ldots\partial x_n^{h_n}}$ if $h_j > 0$ or to the order 0 differential functional $\frac{\partial^0}{x_j^0}$ otherwise, where $h = \sum_{i=1}^{n} h_i$.

Definition 3 ([DLZ11, Definition 1]). *Let $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ such that $F^{-1}(0)$ is a finite set and let $a \in \mathbb{R}^n$ be a solution of the system $S = \{F = 0\}$. Consider the ascending chain of dual spaces $D_a^0[F] \subseteq D_a^1[F] \subseteq \ldots D_a^h[F] \subseteq \ldots$. If there exists an integer α such that $D_a^\alpha[F] = D_a^{\alpha+1}[F]$, then the dimension of the vector space $D_a^\alpha[F]$ is called the multiplicity of a in the system S. If such an α does not exist, the multiplicity is, by convention, infinite.*

For polynomial systems, the two definitions are equivalent [DLZ11, Theorem 2] and in addition the following proposition shows that algebraic tools can be used in the smooth case.

Proposition 4 ([DLZ11, Corollary 3]). *For an integer $k \geqslant n$, let $F = (f_1, \ldots, f_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ and let $a \in \mathbb{R}^n$ be a solution of the system $\{F = 0\}$. Suppose that the multiplicity of a in $\{F = 0\}$ is $m < \infty$, then the intersection multiplicity at a of the polynomial system $\{G = (g_1, \ldots, g_k) = 0\}$ is also m, where g_i is equal to the Taylor expansion of f_i at a up to degree at least m.*

Singularities of plane curves, nodes and ordinary cusps.

Definition 5 ([AGZV12, §17.1]). *For $i \in \{1, 2\}$, let C_i be a plane curve defined in a neighborhood $U_i \subset \mathbb{R}^2$ of p_i by the 0-set of a smooth function f_i. The pairs (p_1, C_1) and (p_2, C_2) are equivalent, and thus define the same plane curve singularity, if there exists a diffeomorphism φ from U_1 to U_2 such that $f_1 = f_2 \circ \varphi$ and $\varphi(p_1) = p_2$.*

In particular, a singularity is of type A_k if the curve is locally defined at the origin by the 0-set of the function $x^2 - y^{k+1}$. As important special cases, A_1 is called a node singularity and A_2 is called an ordinary cusp singularity, see Figure 1.

Remark 6. *It is worthy to notice that a curve C is an ordinary cusp at a point p if C can be locally parametrized with (z^2, z^3) and p corresponds to the value $z = 0$. This remark is helpful to characterize ordinary cusps in Section 4.*

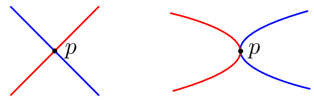

Figure 1: Left: At an A_1 singularity, two branches of the curve intersect transversally. Right: At an A_{2k+1} singularity with $k > 1$, the tangent lines of the two branches at the intersection point coincide.

2.2. Assumptions

Recall that we denote by J_P be the Jacobian matrix of the function P in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Consider the following assumptions:

$\mathcal{A}_1$ For all $q \in \overline{\mathfrak{C}}$, $\operatorname{rank}(J_P(q)) = n-1$. In particular, $\overline{\mathfrak{C}}$ is a smooth curve.[1]

$\mathcal{A}_2$ The set $\mathfrak{L}'_{\mathrm{c}}$ is discrete and does not intersect the boundary of B.

$\mathcal{A}_3$ For all points $p = (\alpha, \beta) \in \pi_{\overline{\mathfrak{C}}}(\overline{\mathfrak{C}})$, the pre-image of p under $\pi_{\overline{\mathfrak{C}}}$ consists of at most two points in $\overline{B}$ counted with multiplicities in the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$.

$\mathcal{A}_4$ The set $\mathfrak{L}'_{\mathrm{n}}$ is discrete and does not intersect the boundary of B.

$\mathcal{A}_5$ The singular points of $\pi_{\mathfrak{C}}(\mathfrak{C})$ are only ordinary cusps or nodes (see Definition 5).

Remark 7. *Regarding Assumption $\mathcal{A}_2$, we will see in Lemma 18 that we can even assume that $\mathfrak{L}'_{\mathrm{c}}$ is empty in the case of a generic curve. However, we are interested in curves where $\mathfrak{L}'_{\mathrm{c}}$ is discrete since the latter case appears in the more specific case of generic silhouette curves (see Section 3.3).*

Lemma 8. *Let $P = (P_1 \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfy Assumption $\mathcal{A}_1$. Let q be in $\overline{\mathfrak{C}}$ such that the multiplicity of the system $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ at q is finite, where $(\alpha, \beta) = \pi_{\overline{\mathfrak{C}}}(q) \in \mathbb{R}^2$. Then, $q \in \mathfrak{L}'_{\mathrm{c}}$ if and only if the multiplicity of the system S at q is at least two.*

Proof. Without loss of generality assume that $q = 0 \in \mathbb{R}^n$.

Sufficiency: Assume that $q \in \mathfrak{L}'_{\mathrm{c}}$. Let $v = (v_1, \dots, v_n)$ be a non-trivial vector of the tangent line of $\overline{\mathfrak{C}}$ at q. Thus, $J_P(q) \cdot v^T = 0$. By the definition of $\mathfrak{L}'_{\mathrm{c}}$ we have $v_1 = v_2 = 0$. Define the differential operator $c = \sum_{i=3}^{n} v_i \frac{\partial}{\partial x_i}$. Notice that $c \cdot P_j = \sum_{i=3}^{n} v_i \frac{\partial P_j}{\partial x_i}(q) = 0$ for all integers $1 \leqslant j \leqslant n-1$ (see [DLZ11, 2.1] for the definition of $c \cdot P_j$). Moreover, by the definition of c and since $v_1 = v_2 = 0$, we have $c \cdot (x_1) = c \cdot (x_2) = 0$. Hence, $c \in D^1_q[S] \setminus D^0_q[S]$. Thus, $\dim(D^1_q) > 1$. Hence, the multiplicity of S at q is at least two.

Necessity: Assume that the multiplicity of S at q is at least two, then $D^0_q[S] \subsetneq D^1_q[S]$. This implies that there exists a non-trivial differential operator $c = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} \in D^1_q[S] \setminus D^0_q[S]$ such that:

[1] Note that the converse is not true as the vertical (double) line defined by $x_1^2 = x_2 = 0$ in $\mathbb{R}^3$ is smooth but the rank of its Jacobian is never full.

(a) We have that $c \cdot P_j = 0$ for all integers $1 \leqslant j \leqslant n-1$ which implies that if we write $v_i = c_i$, with $1 \leqslant i \leqslant n$, the non-trivial vector v is in the tangent space of $\overline{\mathfrak{C}}$ at q.

(b) We have that $c \cdot (x_1) = c \cdot (x_2) = 0$, equivalently, $c_1 = c_2 = 0$. Thus, $v_1 = v_2 = 0$.

The tangent line to the curve at q is thus orthogonal to the (x_1, x_2)-plane. Thus, $q \in \mathfrak{L}'_{\mathrm{c}}$. □

3. Genericity of the assumptions

The key to prove the genericity of our assumptions is Thom's Transversality Theorem. We thus first recall, in Section 3.1, the basics of transversality theory using the notation of Demazure's book [Dem00]. We then prove, in Section 3.2, that all assumptions of Section 2 are satisfied for a generic curve. Finally, in Section 3.3, we consider the special case where the curve is the silhouette of a surface and prove that Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_4$ are generically satisfied in this case.

3.1. Preliminaries

We work with the set of smooth functions $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ with the weak (or compact-open) topology [Dem00, §3.9.2], that is convergence is understood as uniform on compact subsets and for any derivative. A subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is called residual if it contains the intersection of a countable family of dense open subsets. The space $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is a Baire space [Dem00, Proposition 3.9.3], that is, every residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is dense. A property is generic in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ if it is satisfied by a residual subset.

Definition 9 ([Dem00, §3.8.3])**.** *Let $E \simeq \mathbb{R}^n$ and F be two finite-dimensional real vector spaces and let $r \geqslant 0$ be an integer. Let $P^r(E, F)$ be the vector space of polynomial functions of degree at most r from E to F. For an open subset U of E (with respect to the usual topology on E), let $J^r(U, F) = U \times P^r(E, F)$ be the space of jets of order r of functions from U to F. Notice that $J^r(U, F)$ can be identified with an open subset of $\mathbb{R}^N$ for some positive integer N. Let $f : U \to F$ be a smooth function, the jet of order r of f is the function*

$$\begin{aligned} j^r f :\ & U \subset \mathbb{R}^n \to J^r(U, F) \subseteq \mathbb{R}^N \\ & x \mapsto \left(x, f(x), \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x), \frac{\partial^2 f}{\partial x_1 \partial x_2}(x), \ldots, \frac{\partial^r f}{\partial x_n^r}(x) \right). \end{aligned}$$

Let W be a sub-manifold of $J^r(U, F)$. We say that $j^r f$ is transverse to W if for all $a \in U$ either $j^r f(a) \notin W$ or every vector of $\mathbb{R}^N$ can be written as a sum of a vector of $T_{j^r f(a)} W$ and a vector in the image of the function $T_a j^r f$, where $T_{j^r f(a)} W$ is the tangent space of W at $j^r f(a)$ and $T_a j^r f$ is the derivative function of $j^r f$ at a.

Theorem 10 (Thom's Transversality Theorem [Dem00, Theorem 3.9.4])**.** *Let E and F be two finite-dimensional vector spaces with U an open set in E. Let $r \geqslant 0$ be an integer and W be a sub-manifold of $J^r(U, F)$. Then, the set of functions $f \in C^\infty(U, F)$ such that $j^r f$ is transverse to W is a dense residual subset of $C^\infty(U, F)$.*

Proposition 11 ([Dem00, Corollary 3.7.3])**.** *Let U be an open subset of $\mathbb{R}^n$, $N \geqslant 1$ be an integer and W be a sub-manifold of the vector space $\mathbb{R}^N$ of pure co-dimension m. Assume that the smooth function $g : U \to \mathbb{R}^N$ is transverse to W, then $g^{-1}(W)$ is a (possibly empty) sub-manifold of dimension $n - m$.*

The idea of our proofs of genericity of our assumptions is to express each of them as a system of equations in the jet space. When this system defines a manifold, Thom's theorem applies directly to pull back the manifold from the jet space to the ambient space of the curve and obtain the subset where the assumption is satisfied together with its dimension according to Proposition 11. A difficulty occurs when the system does not define a manifold. The following corollary overcomes this difficulty in the special case where the system is defined by analytic functions, in other words the system defines an analytic variety. Such a variety does not need to be a manifold but, using the Whitney stratification theorem [Whi65], the variety is written as a union of manifolds on which Thom's theorem is then applied.

Corollary 12. *Let E and F be two finite-dimensional vector spaces with E of dimension n and U an open set in E. Let $r \geqslant 0$ be an integer and W be an analytic variety of $J^r(U, F)$ with co-dimension larger than n, then for a generic $P \in C^\infty(U, F)$, the pre-image of W under $j^r P$ is empty.*

Proof. Let $W = \bigcup_{i=1}^{m} W_i$ be a Whitney stratification of W, where the W_i's are sub-manifolds. Since $\operatorname{codim}(W) > n$, we have that $\operatorname{codim}(W_i) > n$ for any integer $1 \leqslant i \leqslant m$. Let $\Gamma_i = \{P \in C^\infty(U, F) \mid j^r P \text{ is transverse to } W_i\}$ and $\Gamma = \bigcap_{i=1}^{m} \Gamma_i$. By Theorem 10, Γ_i is residual and so is Γ. Moreover, by Proposition 11, for $P \in \Gamma_i$ the pre-image of W_i under $j^r P$ is empty. Hence, $(j^r P)^{-1}(W) = \bigcup_{i=1}^{m} (j^r P)^{-1}(W_i) = \emptyset$. □

We will also need a refined version of Thom's theorem in a multijet setting, that is for several points in the source space simultaneously. We give the formal definitions of the multijet space and function but we do not restate Theorem 10, Proposition 11 and Corollary 12 that also hold for multijets.

Definition 13 ([Dem00, §3.9.6])**.** *Let U be an open subset of $\mathbb{R}^n$ and $k \geqslant 1$ be an integer. We denote $\Delta_{(k)}(U)$ the subset of U^k consisting of sequences $(a_1, \ldots, a_k)$ of pairwise distinct points of U. For an integer $r \geqslant 0$ and a finite dimensional space F, the k-multijet space of order r, $J^r_{(k)}(U, F)$, is the subset of $J^r(U, F)^k = (U \times P^r(E, F))^k$ consisting of the k-tuples $((a_1, p_1), \ldots, (a_k, p_k))$, with $(a_1, \ldots, a_k) \in \Delta_{(k)}(U)$. Let $f : U \to F$ be a smooth function, the k-multijet of order r of f is the function*

$$\begin{aligned} j^r_{(k)} f :\ & \Delta_{(k)}(U) \to J^r_{(k)}(U, F) \\ & (a_1, \ldots, a_k) \mapsto (j^r f(a_1), \ldots, j^r f(a_k)). \end{aligned}$$

Finally, we gather several technical tools from algebra and analysis.

Proposition 14 ([BV88, Proposition 1.A.1.1])**.** *Let $M(m, n)$ be the vector space of real matrices of size $m \times n$ and r be a positive integer such that $r < \min\{n, m\}$. The determinantal variety, M_r, is the set of matrices in $M(m, n)$ that have rank less than $r + 1$. Then, the following statements hold:*

(a) M_r is an irreducible variety in $M(m, n)$.

(b) M_r is of dimension $r(n + m - r)$.

(c) The singular locus of M_r is M_{r-1}.

Lemma 15 ([Bôc64, §XIV.61 Theorem 1])**.** *Let $n \geqslant 2$ be an integer, $\{x_{ij}\}_{1 \leqslant j,i \leqslant n}$ be a set of n^2 variables and $\mathbb{C}[x_{ij}]_{1 \leqslant j,i \leqslant n}$ be the ring of complex polynomials with the variables $\{x_{ij}\}$. Then, the determinant of the matrix $(x_{ij})_{1 \leqslant i,j \leqslant n}$ is an irreducible polynomial in $\mathbb{C}[x_{ij}]_{1 \leqslant j,i \leqslant n}$.*

Theorem 16 ([Whi43, Theorem 1 & 2])**.** *Let f be an even (resp. odd) smooth function, then there exists a smooth function g such that $f(x) = g(x^2)$ (resp. $f(x) = x \cdot g(x^2)$).*

3.2. Genericity of the assumptions for a curve in $\mathbb{R}^n$

We are going to prove that each assumption in Section 2 is generic. Hence, the combination of these assumptions is also generic since a countable intersection of residual subsets in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is residual.

Lemma 17. *Assumption $\mathcal{A}_1$ is generic.*

Proof. Consider the jet of order 1 of the function $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$:

$$
\begin{aligned}
j^1 P :\ \mathbb{R}^n &\to J^1(\mathbb{R}^n, \mathbb{R}^{n-1}) = \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1)\times n} \\
x &\mapsto (x, P(x), J_P(x)) = (x, y, z).
\end{aligned}
$$

We represent the jet space by the variables $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}^{(n-1)\times n}$. With abuse of notation, we can see the variable z as a $(n-1) \times n$-matrix. Define the variety $W = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1)\times n} \mid y = 0, \operatorname{rank}(z) \leqslant n-2\}$. The variety W is a product of a determinantal variety in $\mathbb{R}^{(n-1)\times n}$ of dimension $n^2 - n - 2$ (by Proposition 14) and a linear space of dimension n in $\mathbb{R}^n \times \mathbb{R}^{n-1}$. Thus, W is a variety of co-dimension $n+1$ in $\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1)\times n}$. Hence, by Corollary 12, there exists a residual subset $\Gamma_1 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, such that for $P \in \Gamma_1$ the pre-image of W under $j^1 P$ is empty. Consequently, for a generic $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ and any $q \in \overline{\mathfrak{C}}$, we have that $q \notin (j^1 P)^{-1}(W) = \emptyset$, thus $\operatorname{rank}(J_P(q)) = n-1$, which is Assumption $\mathcal{A}_1$. □

Lemma 18. *Assumption $\mathcal{A}_2$ is generic. Moreover, generically, the set $\mathfrak{L}'_{\mathrm{c}}$ is empty.*

Proof. We consider the jet of order 1 of the function $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ as is the proof of Lemma 17 with the same notation. Define the matrix $T_1(z)$ (resp. $T_2(z)$) to be the sub-matrix of z obtained by removing the first (resp. second) column. Consider the variety $W \subset J^1(\mathbb{R}^n, \mathbb{R}^{n-1})$ defined by $\{y = 0 \in \mathbb{R}^{n-1}, \det(T_1(z)) = \det(T_2(z)) = 0\}$. Notice that $\mathfrak{L}'_{\mathrm{c}}$ is included in the pre-image of W under $j^1 P$ since $\mathfrak{L}'_{\mathrm{c}}$ is the set of points of the curve $\overline{\mathfrak{C}}$ that are both x_1 and x_2-critical. By Lemma 15, we have that both $\det(T_1(z))$ and $\det(T_2(z))$ are irreducible polynomials. By [CLO92, §9.4 Prop 10], a proper sub-variety of an irreducible variety is of lower dimension, we deduce that the common zero locus of $\det(T_1(z))$ and $\det(T_2(z))$ is of co-dimension at least two. We deduce that $\operatorname{codim}(W) > n$. By Corollary 12, there exists a residual subset $\Gamma_2 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, such that for $P \in \Gamma_2 \cap \Gamma_1$, the pre-image of W under $j^1 P$ is empty and hence $\mathfrak{L}'_{\mathrm{c}}$ is empty, which implies Assumption $\mathcal{A}_2$. □

Lemma 19. *Assumption $\mathcal{A}_3$ is generic.*

Proof. Let us consider the 3-multijet of order 0:

$$j^0_{(3)}P:\ \Delta_{(3)}(\mathbb{R}^n) \to J^0_{(3)}(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^3$$
$$(x, x', x'') \mapsto ((x, P(x)), (x', P(x')), (x'', P(x''))) = ((x, y), (x', y'), (x'', y''))$$

where every element in the jet space $J^0_{(3)}(\mathbb{R}^n, \mathbb{R}^{n-1})$ is of the form $((x, y), (x', y'), (x'', y''))$, where $x = (x_1, \ldots, x_n)$, $x', x'' \in \mathbb{R}^n$ and $y, y', y'' \in \mathbb{R}^{n-1}$. Consider the linear sub-manifold $W = \{x_1 = x'_1 = x''_1, x_2 = x'_2 = x''_2, y = y' = y'' = 0\}$, the co-dimension of W is thus $3n+1$ which is larger than the dimension of the source space $\Delta_{(3)}(\mathbb{R}^n)$ which is $3n$. Thus, by Corollary 12, there exists a residual subset $\Gamma_3 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, such that for $P \in \Gamma_3$, the pre-image of W by $j^0_{(3)}$ is empty, which translates to the fact that there are no pairwise distinct points q, q', q'' in $\overline{\mathfrak{C}}$ such that $\pi_{\overline{\mathfrak{C}}}(q) = \pi_{\overline{\mathfrak{C}}}(q') = \pi_{\overline{\mathfrak{C}}}(q'')$. This is also equivalent to say that the system $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ has at most two distinct solutions (without counting multiplicities) for any $(\alpha, \beta) \in \mathbb{R}^2$.

Using Γ_1, Γ_2 as defined in the proofs of Lemmas 17 & 18 and Γ_3 defined above, we define $\Gamma_4 = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ which is thus a residual set and let P be in Γ_4. Since P is in Γ_3, the system S has at most two distinct solutions. In addition, since P is in $\Gamma_2 \cap \Gamma_1$, one has that $\mathfrak{L}'_{\mathrm{c}}$ is empty and finally together with Lemma 8, since P is in Γ_1, this implies that these solutions have multiplicity exactly 1 is S. For P in the residual set Γ_4, the number of solutions counted with multiplicities of S is thus at most 2, which is Assumption $\mathcal{A}_3$. ☐

Lemma 20. *Assumption $\mathcal{A}_4$ is generic.*

Proof. Let us consider the 2-multijet of order 0 of P:

$$j^0_{(2)}P:\ \Delta_{(2)}(\mathbb{R}^n) \to J^0_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^2$$
$$(x, x') \mapsto ((x, P(x)), (x', P(x'))) = ((x, y), (x', y'))$$

where every element in the jet space $J^0_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-1})$ is of the form $((x, y), (x', y'))$, where $x = (x_1, \ldots, x_n)$, $x' \in \mathbb{R}^n$ and $y, y' \in \mathbb{R}^{n-1}$. Consider the linear sub-manifold $W = \{x_1 = x'_1, x_2 = x'_2, y = y' = 0\}$ of the jet space $J^0_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-1})$. Notice that, $(j^0_{(2)}P)^{-1}(W)$ contains the set $\widehat{\mathfrak{L}}'_{\mathrm{n}} = \{(q_1, q_2) \in \Delta_{(2)}(\mathbb{R}^n) \cap \overline{\mathfrak{C}} \times \overline{\mathfrak{C}} \mid \pi_{\overline{\mathfrak{C}}}(q_1) = \pi_{\overline{\mathfrak{C}}}(q_2)\}$ and $\mathfrak{L}'_{\mathrm{n}}$ is the image of $\widehat{\mathfrak{L}}'_{\mathrm{n}}$ by the projection $(q_1, q_2) \to q_1$. We have $\dim(\Delta_{(2)}(\mathbb{R}^n)) = 2n$ and, since W is linear, its co-dimension is easily computed $\mathrm{codim}(W) = 2(2n-1) - (2 + 2(n-1)) = 2n$. Proposition 11 thus yields that generically $(j^0_{(2)}P)^{-1}(W)$ is a sub-manifold of dimension zero that is a discrete set in $\mathbb{R}^n$, and so is $\mathfrak{L}'_{\mathrm{n}}$.

Now, we prove that, generically, $\mathfrak{L}'_{\mathrm{n}}$ does not intersect the boundary of B. The boundary ∂B of the box B is included in the union of the supporting hyperplanes H_i of its 2^n faces of dimension $n-1$, that is $\partial B = \cup_{i=1}^{i=2^n} H_i$. Define the linear sub-manifold $W_i = \{((x, y), (x', y')) \in W \mid x \in H_i \text{ or } x' \in H_i\}$, notice that this adds one equation to W and thus increases the co-dimension of W by one, thus $\mathrm{codim}(W_i) = 2n+1$. By Corollary 12, we have that generically, the pre-image of W_i under $j^0_{(2)}P$ is empty, which translates to the fact that there is no point of $\mathfrak{L}'_{\mathrm{n}}$ on $\partial B \cap H_i$. This is also true for any i and thus, generically, $\mathfrak{L}'_{\mathrm{n}}$ does not intersect the boundary of B. ☐

For the genericity of Assumption $\mathcal{A}_5$, we first study the singularity types that occur on the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$ under Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$.

Lemma 21. *Under Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, let $q \in \mathfrak{C}$ and $p = \pi_{\mathfrak{C}}(q)$. If $q \notin \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$, then p is a smooth point of the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$.*

Proof. Since $q \notin \mathfrak{L}_{\mathrm{c}}$, the plane projection of $T_q\mathfrak{C}$ is a line, or equivalently, the derivative $T_q\pi_{\mathfrak{C}}$ of $\pi_{\mathfrak{C}}$ at q is injective. Thus, $\pi_{\mathfrak{C}}$ is an immersion at q ([Dem00, Definition 2.9.3]). Hence, for a small enough neighbourhood U_0 of q in $\mathbb{R}^n$, we have that $\pi_{\mathfrak{C}}$ restricted to $V = U_0 \cap \mathfrak{C}$ is embedding (see [Dem00, Proposition 2.9.6]). We are going to prove that, assuming that U_0 is small enough, the curve $\pi_{\mathfrak{C}}(\mathfrak{C})$ has exactly one branch around $\pi_{\mathfrak{C}}(q)$ which implies that $\pi_{\mathfrak{C}}(\mathfrak{C})$ is smooth at $\pi_{\mathfrak{C}}(q)$ since $\mathfrak{C}$ is smooth at q by Assumption $\mathcal{A}_1$.

To prove this claim, assume that there exists an open subset U_0' in $\mathbb{R}^n$ such that the set $V' = U_0' \cap \mathfrak{C}$ and V are disjoint, but $\pi_{\mathfrak{C}}(q)$ is in the closure of $\pi_{\overline{\mathfrak{C}}}(V')$. Let q_k be a sequence of points in V' such that $\pi_{\mathfrak{C}}(q_k)$ converges to $\pi_{\mathfrak{C}}(q)$. Since $\overline{B}$ is compact, there exists a convergent sub-sequence of q_k that has a limit q' in $\overline{B}$. Notice that $\pi_{\overline{\mathfrak{C}}}(q') = \pi_{\mathfrak{C}}(q)$ by the continuity of $\pi_{\overline{\mathfrak{C}}}$. Hence, q, q' are both in $\mathfrak{L}_{\mathrm{n}}'$. However, since $q \notin \mathfrak{L}_{\mathrm{n}}$, we must have that $q' \notin B$. Hence, q' is in the boundary of B which contradicts Assumption $\mathcal{A}_4$. Hence, the curve $\pi_{\mathfrak{C}}(\mathfrak{C})$ has exactly one smooth branch around $\pi_{\mathfrak{C}}(q)$ which concludes the proof. $\square$

Lemma 22. *Under Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, if $q \in \mathfrak{L}_{\mathrm{n}}$, then $\pi_{\mathfrak{C}}(q)$ is a singular point of the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$. More precisely, either $\pi_{\mathfrak{C}}(q)$ is of type A_{2k+1}^- with $k \geqslant 0$, or there exists a non-null smooth function g defined in a neighbourhood of $0 \in \mathbb{R}$ with $\operatorname{ord}(g) = \infty$ such that $(\pi_{\mathfrak{C}}(\mathfrak{C}), \pi_{\mathfrak{C}}(q))$ is equivalent to the curve defined by $x^2 - g(y^2) = 0$ at the origin.*

Proof. Let $p = \pi_{\mathfrak{C}}(q)$ such that $\pi_{\mathfrak{C}}^{-1}(p) = \{q, q'\}$ and denote C the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$. Without loss of generality, one can assume that $p = (0, 0)$. By Assumptions $\mathcal{A}_2$ and $\mathcal{A}_4$, there exists a neighbourhood $N \subseteq C$ of p such that $\pi_{\mathfrak{C}}^{-1}(N)$ is a union of two smooth (Assumption $\mathcal{A}_1$) open subsets of $\mathfrak{C}$ such that q is on one branch and q' on the other, and $\pi_{\mathfrak{C}}$ restricted to $\pi_{\mathfrak{C}}^{-1}(N) \setminus \{p, q'\}$ is an embedding. The projection of these two smooth branches are thus two smooth curves in the plane. Let these two smooth plane branches be defined by the zero sets of the smooth functions f_1 and f_2 in $C^\infty(\mathbb{R}^2, \mathbb{R})$. Let u (resp. u') be a non-zero tangent vector of $\mathfrak{C}$ at q (resp. q') and v (resp. v') be its projection in $\mathbb{R}^2$. We distinguish two cases:

(a) The vectors v and v' are independent in $\mathbb{R}^2$. Thus, v and v' give rise to a local coordinate system (x, y) in a neighbourhood of p in $\mathbb{R}^2$. The vector v being tangent to the zero set of f_1, one has $\frac{\partial f_1}{\partial x}(p) = 0$ and $\frac{\partial f_1}{\partial y}(p) \neq 0$. By the implicit function theorem [Dem00, Corollary 2.7.3.], we deduce that there exists a real smooth function h_1 such that $y = x^2 \cdot h_1(x)$ is a local parametrization of the zero set of f_1. Similarly, there exists a smooth function h_2 such that $x = y^2 \cdot h_2(y)$ is a local parametrization of the zero set of f_2. Thus $(x, y) \in N$ iff $f(x, y) = f_1(x, y) f_2(x, y) = 0$ iff $(y - x^2 \cdot h_1(x))(x - y^2 \cdot h_2(y)) = 0$, equivalently, $[y - x - x^2 \cdot h_1(x) + y^2 \cdot h_2(y)]^2 - [y + x - x^2 \cdot h_1(x) - y^2 \cdot h_2(y)]^2 = 0$. The change of coordinates $X = y - x + x^2 \cdot h_1(x) + y^2 \cdot h_2(y)$ and $Y = y + x + x^2 \cdot h_1(x) - y^2 \cdot h_2(y)$ is a diffeomorphism since indeed $\det(J_{x,y}(X, Y))_p \neq 0$). Then, the local equation of the curve C at p is of the form $X^2 - Y^2$ with these new coordinates, which means that p is a A_1^- or node singularity.

(b) The vectors v and v' are co-linear. Then, choose $v'' \in T_p\mathbb{R}^2$ linearly independent from v, the vectors v, v'' give rise to a coordinate system (x, y) at p. In this coordinate system, we thus have $\frac{\partial f_1}{\partial x}(p) = \frac{\partial f_2}{\partial x}(p) = 0$, $\frac{\partial f_1}{\partial y}(p) \neq 0$ and $\frac{\partial f_2}{\partial y}(p) \neq 0$. By the implicit function theorem, there exist smooth functions h_1 and h_2 such that locally $f(x, y) = 0$ if and only if $(y - x^2 \cdot h_1(x))(y - x^2 \cdot h_2(x)) = 0$. The last equality is equivalent to $(2y - x^2(h_1(x) + h_2(x)))^2 - x^4(h_1(x) - h_2(x))^2 = 0$. Assumption $\mathcal{A}_4$ ensures that the projections of the two branches have only one common point, such that $h_1(x) - h_2(x)$ does not vanish identically. We recognize two cases:

   (i) $\mathrm{ord}(h_1(x) - h_2(x)) = k \leqslant \infty$, then $h_1(x) - h_2(x) = x^k \cdot u$ with $u(p) \neq 0$ and without loss of generality, assume that $u(p) > 0$. The change of coordinates $X = 2y - x^2(h_1(x) + h_2(x))$ and $Y = x \cdot u^{\frac{1}{2+k}}$ is a diffeomorphism (notice that indeed $u^{\frac{1}{2+k}}$ is a smooth function around p). Then, the local equation of the curve C at p is of the form $X^2 - Y^{(2k+3)+1}$ with these new coordinates, which means that p is a singularity of type A^-_{2k+3}.

   (ii) $\mathrm{ord}(h_1(x) - h_2(x)) = \infty$. Since the function $x^4(h_1(x) - h_2(x))^2$ is even, by Theorem 16, there exists a smooth function g such that $x^4(h_1(x) - h_2(x))^2 = g(x^2)$. Thus, taking the diffeomorphism $X = 2y - x^2(h_1(x) + h_2(x))$ and $Y = x$, we get the second case of the claim.

□

The next definition and lemma are technical tools for proving the genericity of Assumption $\mathcal{A}_5$.

Definition 23. *Consider $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfying Assumption $\mathcal{A}_1$ and recall that we denote by $J_P(q)$ the Jacobian matrix of P at the point q. We define the $(n-1) \times (n-2)$ sub-matrix $M_P(q)$ obtained by removing the first two columns of $J_P(q)$ and the $(n-1) \times 2$ sub-matrix $N_P(q)$ formed by the first two columns of $J_P(q)$. Let $q_1, q_2 \in \mathfrak{C}$, we define the square matrix of size $2n-2$,* $M(q_1, q_2) = \begin{pmatrix} N_P(q_1) & 0 & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & 0 \end{pmatrix}$.

Lemma 24. *Using the same assumption and notation as in Definition 23, let q_1 and q_2 be distinct points of $\mathfrak{C}$ with $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, then $M(q_1, q_2)$ is invertible if and only if none of q_1 or q_2 is in $\mathfrak{L}_{\mathrm{c}}$ and the plane projections of the tangent lines of $\mathfrak{C}$ at q_1 and q_2 do not coincide.*

Proof. We prove the converse statement using

$$\begin{aligned} \det(M(q_1, q_2)) = 0 \iff & \text{There exist } \alpha \in \mathbb{R}^2 \text{ and } \beta, \gamma \in \mathbb{R}^{n-2} \text{ such that the vector} \\ & x = (\alpha, \beta, \gamma) \text{ is not trivial and that } M(q_1, q_2) \cdot x^T = 0. \\ \iff & (\alpha, \beta) \text{ and } (\alpha, \gamma) \text{ are in the tangent lines } T_{q_1}\mathfrak{C} \text{ and } T_{q_2}\mathfrak{C} \text{ respectively} \\ & \text{and at least one of them is not trivial.} \end{aligned}$$

The last statement can be split in two cases:

- α is not trivial which is equivalent to say that the plane projections of $T_{q_1}\mathfrak{C}$ and $T_{q_2}\mathfrak{C}$ are both generated by α and coincide.

- $\alpha = (0,0)$ which is equivalent to β or γ is not trivial, which is equivalent to $T_{q_2}\mathfrak{C}$ or $T_{q_1}\mathfrak{C}$ projects to a point in the plane, which is equivalent to q_1 or q_2 is in $\mathfrak{L}_{\mathrm{c}}$.

$\square$

Corollary 25. *Assumption $\mathcal{A}_5$ is generic.*

Proof. Let B be an open n-box. Recall that generically $\mathfrak{L}'_{\mathrm{c}}$ (and hence $\mathfrak{L}_{\mathrm{c}}$) is empty (Lemma 18). Hence, it is enough to prove that for a generic $P \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-1})$, the singular points of $\pi_{\mathfrak{C}}(\mathfrak{C})$ are only nodes (recall that by Lemma 21, under the generic assumptions $\mathcal{A}_1$, $\mathcal{A}_2$,$\mathcal{A}_3$ and $\mathcal{A}_4$, the points in $\mathfrak{C} \setminus (\mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}})$ project to smooth points).

Let Γ_0 be the set of $P \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-1})$ such that P satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. The previous lemmas of this section show that Γ_0 is residual in $C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let us consider the 2-multijet of order 1 of P:

$$\begin{aligned} j^1_{(2)}P :\ & \Delta_{(2)}(\mathbb{R}^n) \to J^1_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-1}) \subseteq (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1)\times n})^2 \\ & (x, x') \mapsto ((x, P(x), J_P(x)), (x', P(x'), J_P(x'))) = ((x, y, z), (x', y', z')) \end{aligned}$$

Let s, s' (resp. r, r') be the sub-matrices of z, z' respectively obtained by removing the first two columns (resp. obtained by the first two columns). Define the matrix $M = \begin{pmatrix} r & 0 & s \\ r' & s' & 0 \end{pmatrix}$ and the variety

$$W = \{((x, y, z), (x', y', z')) \in (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1)\times n})^2 \mid y = y' = 0, x_1 = x'_1, x_2 = x'_2, \det(M) = 0\}.$$

The variety W is a product of a determinantal variety and a linear space, thus its co-dimension is $\operatorname{codim}(W) \geqslant 2n + 1 > 2n = \dim(\Delta_{(2)}(\mathbb{R}^n))$. Hence, by Corollary 12, there exists a residual subset Γ'_0 in $C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-1})$ such that for all $P \in \Gamma'_0$, the pre-image of W under $j^1_{(2)}P$ is empty.

Let then P be in the residual set $\Gamma_0 \cap \Gamma'_0$. By Lemma 24 and since $\mathfrak{L}_{\mathrm{c}}$ is empty, we deduce that for distinct $q_1, q_2 \in \mathfrak{C}$ with $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, the plane projections of the lines $T_{q_1}\mathfrak{C}$ and $T_{q_2}\mathfrak{C}$ intersect transversely if and only if $j^1_{(2)}((q_1, q_2)) \notin W$. Finally, by Lemma 22 (Step (a) of the proof), we deduce that $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ is a node in $\pi_{\mathfrak{C}}(\mathfrak{C})$. $\square$

3.3. Genericity of the assumptions for the silhouette of a surface in $\mathbb{R}^n$

In this section, we focus on the special case of silhouette curves of surfaces in $\mathbb{R}^n$. For an open n-box B and $\widetilde{P}$ in $C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that $S = \widetilde{P}^{-1}(0)$ is a smooth 2-sub-manifold in $\mathbb{R}^n$, the silhouette of $\widetilde{P}$ is the set of points q of this surface S such that the projection (with respect to a fixed direction) of the tangent plane T_qS to $\mathbb{R}^2$ is not surjective. We prove that Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ & $\mathcal{A}_4$ are satisfied for a generic silhouette, and we only conjecture that Assumptions $\mathcal{A}_3$ & $\mathcal{A}_5$ also hold generically. We start by formalizing algebraically the definition of the silhouette curve.

Definition 26. *For an integer $n \geqslant 3$, let $\widetilde{P} = (P_1, \ldots, P_{n-2}) \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-2})$. Define the smooth function $P_{n-1} = \det\left(\left(\frac{\partial P_i}{\partial x_j}\right)_{\substack{1\leqslant i \leqslant n-2 \\ 3 \leqslant j \leqslant n}}\right)$ and $P = (P_1, \ldots, P_{n-1})$. We define the curve $\mathfrak{C}$ (and $\overline{\mathfrak{C}}$) as in Section 2 and call it the silhouette of $\widetilde{P}$.*

Proposition 27. *For a generic $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption $\mathcal{A}_1$.*

Proof. Consider the jet of order 1 of $\widetilde{P}$:

$$\begin{aligned} j^1\widetilde{P} :\ \mathbb{R}^n &\to J^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n} \simeq \mathbb{R}^{n^2-2} = \mathbb{R}^N \\ x &\mapsto (x, \widetilde{P}(x), J_{\widetilde{P}}(x)) = (x, y, z). \end{aligned}$$

We represent the jet space by the vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-2}$ and the $((n-2)\times n)$-matrix $z \in \mathbb{R}^{(n-2)\times n}$. Let $T(z)$ denote the sub-matrix obtained by removing the first two columns of z. Define the variety $W = \{y = 0, \det(T(z)) = 0\} = \{y = 0, \operatorname{rank}(T(z)) \leqslant n-3\}$ in $\mathbb{R}^N$. According to Proposition 14, $W = Reg(W) \cup Sing(W)$ where $Reg(W)$ (resp. $Sing(W)$) is the set of smooth (resp. singular) points in W and

$$Reg(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \operatorname{rank}(T(z)) = n-3\}$$

$$Sing(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \operatorname{rank}(T(z)) < n-3\}.$$

In addition, Proposition 14 yields that $Reg(W)$ is a manifold of co-dimension $n-1$ and $Sing(W)$ is a variety of co-dimension $n+2$. Since the co-dimension of $Sing(W)$ is larger than that of the source space, Corollary 12 implies that, generically, $(j^1\widetilde{P})^{-1}(Sing(W)) = \emptyset$. One thus have $(j^1\widetilde{P})^{-1}(W) = (j^1\widetilde{P})^{-1}(Reg(W))$.

Consider the function

$$\begin{aligned} \varphi :\ \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n} &\to \mathbb{R}^{n-2} \times \mathbb{R} \\ \chi = (x, y, z) &\mapsto (y, \det(T(z))), \end{aligned}$$

such that $\varphi^{-1}(0) = W$. Its Jacobian matrix is $J_\varphi = \begin{pmatrix} 0_{(n-2)\times n} & I_{(n-2)\times(n-2)} & 0_{(n-2)\times(n-2)n} \\ 0_{1\times(n)} & 0_{1\times(n-2)} & v(z) \end{pmatrix}$, where $0_{k_1\times k_2}$ (resp. $I_{k_1\times k_2}$) is the zero (resp. identity) matrix of size $k_1 \times k_2$ and the vector $v(z)$ is the adjugate matrix of $T(z)$ written as the concatenation of its lines: $v(z) = (Adj^{ij}(T(z)))_{\substack{1\leqslant i\leqslant n-2 \\ 3\leqslant j\leqslant n}} \in \mathbb{R}^{(n-2)^2}$. Let $\chi = (x, y, z) \in Reg(W)$, then $\operatorname{rank}(T(z)) = n-3$, thus there exists a pair (i, j) such that $Adj^{ij}(T(z)) \neq 0$. Hence, the vector $v(z)$ is non-trivial and $J_\varphi(\chi)$ has full rank $n-1$. The function φ is thus a submersion on $Reg(W)$.

Theorem 10 yields that, generically, $j^1\widetilde{P}$ is transverse to the manifold $Reg(W)$. Together with the fact that φ is a submersion on $Reg(W)$, [GG73, Lemma II.4.3 (p.52)] implies that $P = \varphi \circ j^1\widetilde{P}$ is a submersion on $(j^1\widetilde{P})^{-1}(Reg(W)) = (j^1\widetilde{P})^{-1}(W) = (j^1\widetilde{P})^{-1}(\varphi^{-1}(0)) = (\varphi \circ j^1\widetilde{P})^{-1}(0) = P^{-1}(0) = \mathfrak{C}$. In other words, J_P has full rank $n-1$ on $\mathfrak{C}$, which is Assumption $\mathcal{A}_1$. $\square$

Proposition 28. *For a generic $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption $\mathcal{A}_2$.*

Proof. First we prove that, generically, $\mathfrak{L}'_c$ is discrete. For any $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ consider $j^2\widetilde{P} : \mathbb{R}^n \to J^2(\mathbb{R}^n, \mathbb{R}^{n-2}) \subset \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n} \times \mathbb{R}^{n^2(n-2)} = \mathbb{R}^N$. Assume that every element in $\mathbb{R}^N$ is represented as (x, y, z, h), where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-2}$, $z \in \mathbb{R}^{(n-2)\times n}$ and $h \in \mathbb{R}^{n^2(n-2)}$. With abuse of notation we can consider z as a $((n-2)\times n)$-matrix. Let $T(z)$ denote the matrix obtained by removing the first two columns of z. The Jacobian matrix J_P is a function of the derivatives $(\frac{\partial P_i}{\partial x_j}, \frac{\partial^2 P_l}{\partial x_k \partial x_s})_{\substack{1\leqslant i,l\leqslant n-2\\ 1\leqslant j,k,s\leqslant n}}$, it can thus be seen in the jet space as a function of z and h, $J_P(z,h)$. Define the matrix $T_1(z,h)$ (resp. $T_2(z,h)$) to be the sub-matrix of $J_P(z,h)$ obtained by removing the first (resp. second) column. Define the variety $W = \{(x,y,z,h) \mid y = 0 \in \mathbb{R}^{n-2}, \det(T(z)) = \det(T_1(z,h)) = \det(T_2(z,h)) = 0\}$, so that $\mathfrak{L}'_{\mathrm{c}}$ is included in the pre-image of W under $j^2\widetilde{P}$. Let $W_1 = \{(x,y,z,h) \mid y = 0 \in \mathbb{R}^{n-2}, \det(T(z)) = 0\}$, we already showed in the proof of Proposition 27 that W_1 is an irreducible variety of co-dimension $n-1$. In addition, $\det(T_1(z,h))$ does not identically vanish on W_1, thus W is a proper sub-variety of the irreducible variety W_1 and [CLO92, §9.4 Prop 10] implies that $\mathrm{codim}(W) > \mathrm{codim}(W_1) = n-1$.

Now, write $W = Reg(W) \cup Sing(W)$, where $Reg(W)$ (resp. Sing(W)) is the set of smooth (resp. singular) points in W. Recall that $\mathrm{codim}(Sing(W)) > n$ since $Sing(W)$ is a proper closed sub-variety of W [BCR98, Proposition 3.3.14]. By Corollary 12, there exists a residual set $\Gamma' \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that if $\widetilde{P} \in \Gamma'$, then the pre-image of $Sing(W)$ under $j^2\widetilde{P}$ is empty. Define $\Gamma = \{\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2}) \mid j^2\widetilde{P} \text{ is transverse to } Reg(W)\} \cap \Gamma'$. Notice that if $\widetilde{P} \in \Gamma$, then $\mathfrak{L}'_{\mathrm{c}}$ is included in the pre-image of $Reg(W)$ under $j^2\widetilde{P}$. Hence, since $\mathrm{codim}(Reg(W)) = \mathrm{codim}(W) \geqslant n$, we have by Proposition 11 that $\mathfrak{L}'_{\mathrm{c}}$ is a sub-manifold of dimension, at most, zero. Thus, $\mathfrak{L}'_{\mathrm{c}}$ is discrete for all $\widetilde{P} \in \Gamma$. Using Theorem 10 we deduce that Γ is residual.

The proof that $\mathfrak{L}'_{\mathrm{c}}$ does not intersect the boundary of B can be done analogously as in the proof of Lemma 20. □

Proposition 29. *For a generic $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption $\mathcal{A}_4$.*

Proof. Consider the 2-multijet $j^1_{(2)}\widetilde{P} : \Delta_{(2)}(\mathbb{R}^n) \to J^1_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n})^2$ of the function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, where $(\mathbb{R}^n \times \mathbb{R}^{(n-2)\times n} \times \mathbb{R}^{(n-2)\times n})^2$ is described by the coordinates $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^{n-2}$ and $z, z' \in \mathbb{R}^{(n-2)\times n}$. With abuse of notation we can consider z and z' as $((n-2)\times n)$- matrices. Let $T(z)$ (resp. $T(z')$) denote the matrix obtained by removing the first two columns of z (resp. z'). Define the variety W to be the solution set of the system $\{y = y' = 0, x_1 - x'_1 = x_2 - x'_2 = \det(T(z)) = \det(T(z')) = 0\}$. Denote by $Reg(W)$ the regular part of W. By Proposition 14 (a) we deduce that W is of co-dimension $2n$. Using the same argument in the proof of Proposition 27, we deduce that there exists a residual set $\Gamma \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that if $\widetilde{P} \in \Gamma$, then the image of $\Delta_2(\mathbb{R}^n)$ under $j^1_{(2)}\widetilde{P}$ is contained in $Reg(W)$. Moreover, by Proposition 11, we have that $M_P = (j^1_{(2)}\widetilde{P})^{-1}(Reg(W)) = (j^1_{(2)}\widetilde{P})^{-1}(W)$ is a sub-manifold of dimension zero in $\Delta_2(\mathbb{R}^n)$. Notice that $\mathfrak{L}'_{\mathrm{n}}$ is the image of M_P under the projection $(x,x') \to x$. Since M_P is of dimension zero, then so is $\mathfrak{L}'_{\mathrm{n}}$. Thus we have just proven that, if $\widetilde{P} \in \Gamma$, then $\mathfrak{L}'_{\mathrm{n}}$ is a sub-manifold of dimension zero. Hence, $\mathfrak{L}'_{\mathrm{n}}$ is discrete.

The proof that $\mathfrak{L}'_{\mathrm{n}}$ does not intersect the boundary of B can be done analogously as in the proof of Lemma 20. □

Assumption $\mathcal{A}_3$ can be rephrased by the three following assumptions:

$\mathcal{A}_{3(a)}$ There are no pairwise distinct $q, q', q'' \in \overline{\mathfrak{C}}$ such that $\pi_{\mathfrak{C}}(q) = \pi_{\mathfrak{C}}(q') = \pi_{\mathfrak{C}}(q'')$.

$\mathcal{A}_{3(b)}$ $\mathfrak{L}'_{\mathrm{c}} \cap \mathfrak{L}'_{\mathrm{n}} = \emptyset$.

$\mathcal{A}_{3(c)}$ For $q \in \mathfrak{L}'_{\mathrm{c}}$, the multiplicity of the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, (x_1, x_2) = \pi_{\mathfrak{C}}(q)\}$ at q is exactly two.

Using this rephrasing, we next show that Assumptions $\mathcal{A}_{3(a)}$ & $\mathcal{A}_{3(b)}$ generically hold and we leave Assumption $\mathcal{A}_{3(c)}$ as a conjecture.

Proposition 30. *For a generic function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_{3(a)}$ holds.*

Proof. Consider the 3-multijet $j^1_{(3)}\widetilde{P} : \Delta_{(3)}(\mathbb{R}^n) \to J^1_{(3)}(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n})^3$. Assume that every element in $(\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)n})^3$ is of the form $((x, y, z), (x', y', z'), (x'', y'', z''))$, where $x, x', x'' \in \mathbb{R}^n$, $y, y', y'' \in \mathbb{R}^{n-2}$ and $z, z', z'' \in \mathbb{R}^{(n-2)\times n}$. With abuse of notation we can consider z, z' and z'' as $((n-2) \times n)$-matrices. Let $T(z), T(z'), T(z'')$ denote the matrices obtained by removing the first two columns of z, z', z'' respectively. Consider the variety W defined by the equations: $\{x_1 = x'_1 = x''_1, x_2 = x'_2 = x''_2, y = y' = y'' = 0 \in \mathbb{R}^{n-2}, \det(T(z)) = \det(T(z')) = \det(T(z'')) = 0\}$.

Notice that $\dim(\Delta_{(3)}(\mathbb{R}^n)) = 3n < 3n + 1 = \operatorname{codim}(W)$. Hence, by Corollary 12, we have that, generically, the pre-image of W under $j^1_{(3)}\widetilde{P}$ is empty. Hence, there are no pairwise different $q, q', q'' \in \overline{\mathfrak{C}}$ such that $\pi_{\mathfrak{C}}(q) = \pi_{\mathfrak{C}}(q') = \pi_{\mathfrak{C}}(q'')$. $\square$

Proposition 31. *For a generic function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_{3(b)}$ holds.*

Proof. Consider the 2-multijet $j^2_{(2)}\widetilde{P} : \Delta_{(2)}(\mathbb{R}^n) \to J^2_{(2)}(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2)\times n} \times \mathbb{R}^{n^2(n-2)})^2$ of the function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, where $(\mathbb{R}^n \times \mathbb{R}^{(n-2)\times n} \times \mathbb{R}^{(n-2)\times n} \times \mathbb{R}^{n^2(n-2)})^2$ is described by the coordinates $x, x' \in \mathbb{R}^n, y, y' \in \mathbb{R}^{n-2}, z, z' \in \mathbb{R}^{(n-2)\times n}$ and $h, h' \in \mathbb{R}^{n^2(n-2)}$. With abuse of notation we can consider z and z' as $((n-2) \times n)$- matrices. Let $T(z)$ (resp. $T(z')$) denote the matrix obtained by removing the first two columns of z (resp. z'). Define the matrices $T_1(z, h), T_2(z, h)$ as in the proof of Lemma 28 and the variety W to be the solution set of the system $\{y = y' = 0 \in \mathbb{R}^{n-2}, x_1 - x'_1 = x_2 - x'_2 = det(T(z)) = det(T(z') = 0, \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$.

Define varieties $W' = \{(x, y, z, h) \mid y = y' = 0, det(T(z)) = det(T(z')) = 0, x_1 = x'_1, x_2 = x'_2\}$ and $W'' = \{(x, y, z, h) \mid y = y' = 0, \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$. Notice that $W = W' \cap W''$. Moreover, we can find a smooth silhouette curve C that is not an orthogonal line to (x_1, x_2)-plane and that contains two distinct points q, q', with $\pi_{\mathfrak{C}}(q) = \pi_{\mathfrak{C}}(q')$ such that the projection of T_qC (resp. $T_{q'}C$) onto $\mathbb{R}^2$ is injective. Notice that $j^2_{(2)}\widetilde{P}(q, q') \in W' \setminus W''$. Hence, $W' \not\subseteq W''$. Moreover, since W' is the Cartesian product of determinant varieties (which are irreducible by Proposition 14(a)) with linear spaces, we have that W' is also irreducible [BCR98, Theorem 2.8.3 (iii)]. In other words, $W = W' \cap W''$ is a proper sub-variety of the irreducible variety W'. Hence, $\dim(W) = \dim(W' \cap W'') < \dim(W')$, equivalently, $\operatorname{codim}(W) > \operatorname{codim}(W') = 2n$. Hence, by Corollary 12 we have that, generically, the pre-image of W under $j^2_{(2)}\widetilde{P}$ is empty. Since, by Proposition 27, Assumption $\mathcal{A}_1$ (which is necessary to guarantee that $\mathfrak{L}'_{\mathrm{c}}$ is well-defined) is also generic, we imply that, generically, there is no distinct pair $q, q' \in \mathfrak{C}$ such that $\pi_{\mathfrak{C}}(q) = \pi_{\mathfrak{C}}(q')$ and $q \in \mathfrak{L}'_{\mathrm{c}}$, equivalently, $\mathfrak{L}'_{\mathrm{c}} \cap \mathfrak{L}'_{\mathrm{n}} = \emptyset$ which proves the proposition. $\square$

Conjecture 32. *For a generic function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_{3(c)}$ holds.*

Conjecture 33. *For a generic function $\widetilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_5$ holds.*

4. Modelling System

Our goal in this section is to encode the singularities of $\pi_{\mathfrak{C}}(\mathfrak{C})$ by a square and regular (see Definition 50) system so that it is solvable with certified numerical methods. In Section 4.1, we first define this system $\mathrm{Ball}(P)$. In Section 4.2, we then locally parametrize the curve around the points in $\mathfrak{L}_{\mathrm{c}}$ to simplify the computation of $\mathrm{Ball}(P)$ and its Jacobian. In Section 4.3, we determine the necessary and sufficient conditions for this system to be regular.

4.1. Encoding the singular points of the plane projection

Definition 34. *Let y, r be two variables in $\mathbb{R}^{n-2}$ and t be a variable in $\mathbb{R}$. For a smooth function $f : \overline{B} \subset \mathbb{R}^n \to \mathbb{R}$, we define the functions:*

$$S \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0 \end{cases}$$

and

$$D \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ \nabla f(x_1, x_2, y) \cdot (0, 0, r) = \sum\limits_{3}^{n} \frac{\partial f}{\partial x_i} r_i, & \text{for } t = 0. \end{cases}$$

Lemma 35. *If f is a smooth function defined on $\overline{B} \subseteq \mathbb{R}^n$, then both $S \cdot f$ and $D \cdot f$ are smooth functions on the subset*

$$\overline{B}_{\mathrm{Ball}} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geqslant 0, (x_1, x_2, y \pm r\sqrt{t}) \in \overline{B}, \|r\|^2 = 1\}$$

of $\mathbb{R}^{2n-1}$, where $\|r\|$ denotes the Euclidean norm of r.

Proof. On the subset $\overline{B}_{\mathrm{Ball}}$ with $t > 0$, both $S \cdot f(x_1, x_2, y, r, t)$ and $D \cdot f(x_1, x_2, y, r, t)$ are the compositions of smooth functions, hence they are smooth functions.

For a point $X = (x_1, x_2, y, r, t)$ in B_{Ball} with $t = 0$, we will prove that $S \cdot f$ (resp. $D \cdot f$) is a C^s function for an arbitrarily s which implies that $S \cdot f$ (resp. $D \cdot f$) is smooth. First define the function

$$S_0 \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + rt) + f(x_1, x_2, y - rt)], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0. \end{cases}$$

Since $S_0 \cdot f(x_1, x_2, y, r, t)$ is an even smooth function with respect to t, the partial derivatives of $S_0 \cdot f$ with respect to t of odd orders, evaluated at X, are zero. For an integer $s > 0$, by the parametrized Taylor formula without remainder [Dem00, Proposition 4.2.2], there exist smooth functions $a_i(x_1, x_2, y, r)$, with integers $0 \leqslant i < s$ such that $S_0 \cdot f(x_1, x_2, y, r, t) = \sum\limits_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^{2i} + t^{2s} \cdot \phi(x_1, x_2, y, t)$, where $\phi(x_1, x_2, y, t)$ is a smooth function. Notice that $S \cdot f(x_1, x_2, y, r, t) = \sum\limits_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^i + t^s \cdot \phi(x_1, x_2, y, \sqrt{t})$, so that a partial derivative exists up to order s at $t = 0$. Thus, $S \cdot f(x_1, x_2, y, r, t)$ is a C^{s-1} function. This holds for any arbitrarily large s, hence $S \cdot f(x_1, x_2, y, r, t)$ is a C^∞ function.

Now, we prove that $D \cdot f$ is continuous at $X = (x_1, x_2, y, r, 0)$. Let X_i be a sequence that converges to X. To prove that $D \cdot f(X_i)$ converges to $D \cdot f(X)$, it is enough to show that for a sequence t_i that converges to 0, then we have that $D \cdot f(x_1, x_2, y, r, t_n)$ converges to $D \cdot f(X)$. We can assume that $t_i \neq 0$ for all i, so that

$$\begin{aligned}
\lim_{t_i \to 0} D \cdot f(x_1, x_2, y, r, t_i) &= \lim_{t_i \to 0} \frac{1}{2\sqrt{t_i}}[f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y - r\sqrt{t_i})] \\
&= \lim_{t_i \to 0} \frac{1}{2\sqrt{t_i}}[f(x_1, x_2, y + r\sqrt{t_i}) - (f(x_1, x_2, y) - f(x_1, x_2, y)) - f(x_1, x_2, y - r\sqrt{t_i})] \\
&= \lim_{t_i \to 0} \frac{1}{2\sqrt{t_i}}[f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y)] \\
&\quad\quad + \lim_{t_i \to 0} \frac{1}{2\sqrt{t_i}}[f(x_1, x_2, y) - f(x_1, x_2, y - r\sqrt{t_i})] \\
&= \frac{1}{2}\nabla f \cdot (0, 0, r) - \frac{1}{2}\nabla f \cdot (0, 0, -r) \\
&= \nabla f \cdot (0, 0, r).
\end{aligned}$$

We now prove that $D \cdot f$ is smooth at X. Similarly to the proof of the case of $S \cdot f$, since the function $\frac{1}{2}[f(x_1, x_2, y + rt) - f(x_1, x_2, y - rt)]$ is odd with respect to t, there exist smooth functions $b_i(x_1, x_2, y, r)$, for $1 \leqslant i < s$ and $\psi(x_1, x_2, y, r, t)$ such that $\frac{1}{2}[f(x_1, x_2, y+rt) - f(x_1, x_2, y-rt)] = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r)t^{2i+1} + t^{2s+1} \cdot \psi(x_1, x_2, y, t)$. Notice that $D \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r)t^i + t^s \cdot \psi(x_1, x_2, y, \sqrt{t})$, so that a partial derivative exists up to order s at $t = 0$. Thus, $D \cdot f(x_1, x_2, y, r, t)$ is a C^{s-1} function. This holds for any arbitrarily large s, hence $D \cdot f(x_1, x_2, y, r, t)$ is a C^∞ function. □

Theorem 36. *Consider $P = (P_1, \ldots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ that satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. Then, $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ is a solution of the* Ball system

$$\mathrm{Ball}(P) = \begin{cases} S \cdot P_1(X) = \cdots = S \cdot P_{n-1}(X) = 0 \\ D \cdot P_1(X) = \cdots = D \cdot P_{n-1}(X) = 0 \\ \|r\|^2 - 1 = 0 \end{cases} \tag{4.1}$$

if and only if (x_1, x_2) is a singular point of $\pi_{\mathfrak{C}}(\mathfrak{C})$ (see Definition 34 for the notation $S \cdot P_i$ and $D \cdot P_i$).

We postpone the proof of Theorem 36 to the end of Section 4.2. As a first step, we study a mapping from the solutions of the ball system to pairs of points on the curve $\mathfrak{C}$.

Definition 37. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Define $\widehat{\mathfrak{L}}_{\mathrm{n}}$ to be the set of pairs (q_1, q_2) with $q_1, q_2 \in \mathfrak{C}$, $q_1 \neq q_2$ and $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, also define $\widehat{\mathfrak{L}}_{\mathrm{c}}$ to be the set of pairs (q_1, q_1) with $q_1 \in \mathfrak{L}_{\mathrm{c}}$, and let $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}_{\mathrm{n}} \cup \widehat{\mathfrak{L}}_{\mathrm{c}}$.*

Lemma 38. *Consider $P = (P_1, \ldots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ and let $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$, with $\|r\| = 1$. Assume that P satisfies Assumption $\mathcal{A}_1$. Then X is a solution of* $\mathrm{Ball}(P)$ *if and only if for the points $q_1 = (x_1, x_2, y + r\sqrt{t})$ and $q_2 = (x_1, x_2, y - r\sqrt{t})$, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_{\mathrm{n}}$, or in $\widehat{\mathfrak{L}}_{\mathrm{c}}$ with $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ in $T_{q_1}\mathfrak{C}$.*

Proof. Note that, by Assumption $\mathcal{A}_1$, the tangent space to the curve at any of its points is well defined and is a line. First, assume that X is a solution of $\mathrm{Ball}(P)$. We consider two cases:

(a) If $t > 0$, then since $r \neq 0 \in \mathbb{R}^{n-2}$ we have that $q_1 \neq q_2$. Moreover, since $S \cdot P_i(X) = D \cdot P_i(X) = 0$ for all $i \in \{1, \dots, n-1\}$, we deduce that $P_i(q_1) = P_i(q_2) = 0$, thus $q_1, q_2 \in \mathfrak{C}$. Moreover, since $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2) = (x_1, x_2)$ we have $q_1, q_2 \in \mathfrak{L}_{\mathrm{n}}$. Thus, $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$.

(b) If $t = 0$, then $q_1 = q_2$. First, $P_i(q_1) = S \cdot P_i(X) = 0$, for all indices $i \in \{1, \dots, n-1\}$, hence $q_1 \in \mathfrak{C}$. Moreover, we have $0 = D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r)$, for all $i \in \{1, \dots, n-1\}$, equivalently, $J_P(q_1) \cdot (0, 0, r)^T = 0 \in \mathbb{R}^{n-1}$, i.e., we have $(0, 0, r) \in T_{q_1}\mathfrak{C}$. Thus, $q_1 \in \mathfrak{L}_{\mathrm{c}}$ and hence, $(q_1, q_1) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$.

Now, let us prove the other direction:

(a) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$, then $q_1 \neq q_2$ and $t \neq 0$. Also, since $q_1, q_2 \in \mathfrak{C}$, we can write that $S \cdot P_i(X) = \frac{1}{2}(P_i(q_1) + P_i(q_2)) = 0$, and $D \cdot P_i(X) = \frac{1}{2\sqrt{t}}(P_i(q_1) - P_i(q_2)) = 0$, for all $i \in \{1, \dots, n-1\}$. Thus, X is a solution of $\mathrm{Ball}(P)$.

(b) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$ and $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ is in $T_{q_1}\mathfrak{C}$, one has $q_1 = q_2 \in \mathfrak{L}_{\mathrm{c}} \subseteq \mathfrak{C}$, and $t = 0$. Moreover, for all $i \in \{1, \dots, n-1\}$ we have $S \cdot P_i(X) = P_i(q_1) = 0$ and since $(0, 0, r) \in T_{q_1}\mathfrak{C}$, we can equivalently write $D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r) = 0$. Thus, X is a solution of $\mathrm{Ball}(P)$.

□

Definition 39. *Let* $\mathrm{Sol}_{\mathrm{Ball}(P)}$ *be the solution set of* $\mathrm{Ball}(P)$. *Define the function* Ω_P *from* $\mathrm{Sol}_{\mathrm{Ball}(P)}$ *to* $\widehat{\mathfrak{L}}$ *that sends* $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ *to the ordered pair* $q_1 = (x_1, x_2, y + r\sqrt{t})$ *and* $q_2 = (x_1, x_2, y - r\sqrt{t})$. *Notice that the function* Ω_P *is well-defined by Lemma 38.*

Lemma 40. *If* $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ *satisfies Assumption* $\mathcal{A}_1$, *then* Ω_P *is surjective.*

Proof. For any pair $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$ we have that the point $X = (\frac{1}{2}(q_1 + q_2), \frac{\Pi_{\mathfrak{C}}(q_1 - q_2)}{\|q_1 - q_2\|}, \frac{1}{4}\|q_1 - q_2\|^2) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}$ is a solution of $\mathrm{Ball}(P)$, where $\Pi_{\mathfrak{C}}(q_1 - q_2)$ is the vector in $\mathbb{R}^{n-2}$ obtained by omitting the first two coordinates (which are zeros) from $q_1 - q_2$. Note that $\Omega_P(X) = (q_1, q_2)$. If the pair (q_1, q_1) is in Lch, we define r in the following way, we take a unit vector $v \in T_{q_1}\mathfrak{C}$ (the first two coordinates of v are zeros since $q_1 \in \mathfrak{L}_{\mathrm{c}}$). We set r to be $\Pi_{\mathfrak{C}}(v)$. Again $X = (q_1, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ is a solution of $\mathrm{Ball}(P)$, with $\Omega_P(X) = (q_1, q_1)$. Thus, Ω_P is surjective. □

Remark 41. *Notice that if* $X = (x_1, x_2, y, r, t)$ *is in* $\mathrm{Sol}_{\mathrm{Ball}(P)}$, *then* $\Omega_P(X) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$ *(resp.* $\Omega_P(X) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$*) if and only if* $t \neq 0$ *(resp.* $t = 0$*).*

Remark 42. *Preserving the notation in Lemma 38, notice that if* $X = (x_1, x_2, y, r, t)$ *is a solution of* $\mathrm{Ball}(P)$, *then* $X' = (x_1, x_2, y, -r, t)$ *is another solution. Moreover, both solutions characterize the same unordered pair* $\Omega_P(X) = \Omega_P(X') = (q_1, q_2)$. *We call* X *and* X' *twin solutions. An alternative would have been to take* r *in a projective space instead of the sphere to identify these twin solutions.*

Example 43. *Let* $n = 3$ *and* $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \in [-2, 2]\}$. *Define* $P_1(x_1, x_2, x_3) = x_1 - (x_3 - 1)^3$, $P_2(x_1, x_2, x_3) = x_2 - (x_3 - 1)^2$ *and* $P = (P_1, P_2)$. *The Jacobian matrix of* P *has full rank over* $\mathfrak{C}$, *thus Assumption* $\mathcal{A}_1$

is satisfied. The set $\mathfrak{L}_{\mathrm{n}}$ is empty since $\pi_{\mathfrak{C}}$ is injective over $\mathfrak{C}$, hence Assumption $\mathcal{A}_4$ is satisfied. The only point of $\mathfrak{C}$ with a tangent line orthogonal to the (x_1, x_2)-plane is $q_1 = (0,0,1)$, thus $\mathfrak{L}_{\mathrm{c}} = \{q_1\}$ and Assumption $\mathcal{A}_2$ is satisfied. By Lemma 46, the multiplicity of the system $\{P = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$ at its unique solution q_1 is $\min\{\mathrm{ord}_1((x_3-1)^3), \mathrm{ord}_1((x_3-1)^2)\} = \min\{3,2\} = 2$ *(*ord *is defined in Definition 1). Moreover, for any point $q_0 \in \mathfrak{C}$ different from q_1, the multiplicity of the corresponding system at its unique solution q_0 is one, thus P satisfies Assumption $\mathcal{A}_3$. The system* $\mathrm{Ball}(P)$*:*

$$\begin{cases} x_1 - 3r^2ty + 3r^2t - y^3 + 3y^2 - 3y + 1 = 0 \\ x_2 - r^2t - y^2 + 2y - 1 = 0 \\ -r^3t - 3ry^2 + 6ry - 3r = 0 \\ -2ry + 2r = 0 \\ r^2 - 1 = 0 \end{cases} \tag{4.2}$$

has two twin solutions $X = (0,0,1,1,0)$ *and* $X' = (0,0,1,-1,0)$ *in* $B_{\mathrm{Ball}(P)} \subset \mathbb{R}^{2\cdot 3-1} = \mathbb{R}^5$ *such that* $\Omega_P(X) = \Omega_P(X') = (q_1, q_1) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$.

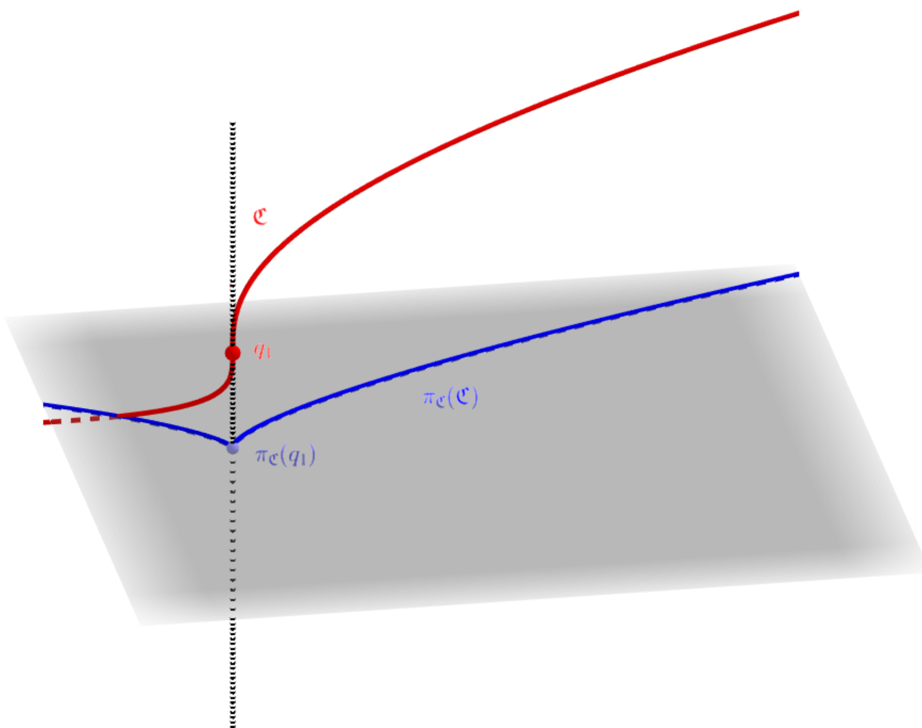

Figure 2: The curve $\mathfrak{C}$ (red) and its plane projection $\pi_{\mathfrak{C}}(\mathfrak{C})$ (blue) of Example 43 displaying a cusp singularity.

Example 44. *Let B be defined as in Example 43. Define the functions $P_1(x_1, x_2, x_3) = x_1 - (x_3^2 - 1)$, $P_2(x_1, x_2, x_3) = x_2 - (x_3^3 - x_3)$ and $P = (P_1, P_2)$. The Jacobian matrix of P has full rank over $\mathfrak{C}$, thus Assumption $\mathcal{A}_1$ is satisfied. Moreover, the set $\mathfrak{L}_{\mathrm{c}}$ is empty and $\mathfrak{L}_{\mathrm{n}} = \{q_1, q_2\}$, with $q_1 = (0, 0, 1), q_2 = (0, 0, -1)$, i.e., Assumptions $\mathcal{A}_2$ and $\mathcal{A}_4$ are satisfied. The multiplicity of the system $\{P = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$ at both q_1, q_2 is equal to one, thus Assumption $\mathcal{A}_3$ is also satisfied. The system* $\mathrm{Ball}(P)$*:*

$$\begin{cases} x_1 - r^2 t - y^2 + 1 = 0 \\ x_2 - r^2 t y - y^3 + y = 0 \\ -2ry = 0 \\ -r^3 t - 3ry^2 + r = 0 \\ r^2 - 1 = 0 \end{cases} \tag{4.3}$$

has two twin solutions $X = (0, 0, 0, 1, 1)$ and $X' = (0, 0, 0, -1, 1)$ in $\mathbb{R}^5$ such that $\Omega_P(X) = \Omega_P(X') = (q_1, q_2) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$.

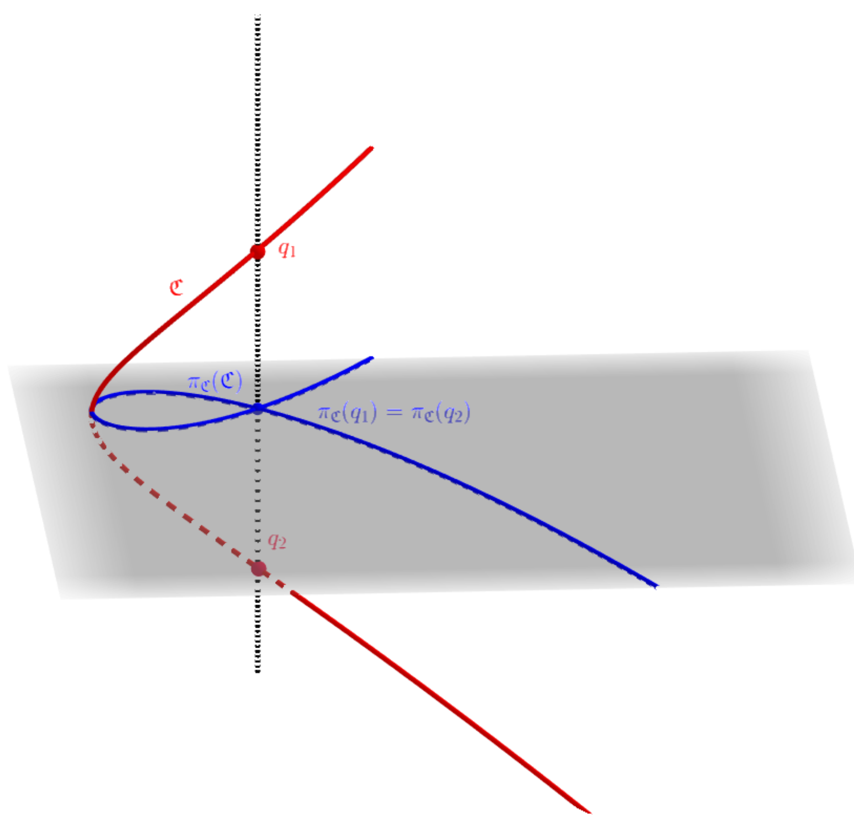

Figure 3: The curve $\mathfrak{C}$ (red) and its plane projection $\pi_{\mathfrak{C}}(\mathfrak{C})$ (blue) of Example 44 displaying a node singularity.

4.2. The characterization of $\mathfrak{C}$ around the points in $\mathfrak{L}_{\mathrm{c}}$

In this section, we are going to locally parametrize P around the points in $\mathfrak{L}_{\mathrm{c}}$. This parametrization will ease the computation of $\mathrm{Ball}(P)$ and its Jacobian in Section 4.3.

Lemma 45. *Let $P \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let $q \in \mathfrak{L}_{\mathrm{c}}$ such that Assumption $\mathcal{A}_1$ is satisfied in a neighbourhood of q in B. Without loss of generality one can assume $q = 0 \in \mathbb{R}^n$. Then there exist an invertible matrix M of size $(n-1)\times(n-1)$ of smooth functions in a neighbourhood of q and smooth functions $f_1, f_2, f_3, \ldots, f_{n-1}$ defined in a neighbourhood of $0 \in \mathbb{R}$, such that:*

$$\begin{pmatrix} x_1 - f_1(x_n) \\ x_2 - f_2(x_n) \\ x_3 - f_3(x_n) \\ \cdots \\ \cdots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \cdots \\ \cdots \\ P_{n-1} \end{pmatrix}, \tag{4.4}$$

with $\min\{\mathrm{ord}(f_1(x_n)), \mathrm{ord}(f_2(x_n))\} > 1$ (ord is defined in Definition 1).

Proof. Since $\mathrm{rank}(J_P(q)) = n-1$ (Assumption $\mathcal{A}_1$), there exists $k \in \{1, \ldots, n\}$ such that $\det(M_k(q)) \neq 0$, where M_k is the minor of J_P obtained by removing the k-th column. Notice that $k \notin \{1, 2\}$, since $q \in \mathfrak{L}_{\mathrm{c}}$ implies that $\det(M_1(q)) = \det(M_2(q)) = 0$. Without loss of generality, we assume that $k = n$. Using the implicit function theorem [Corollary 2.7.3][Dem00], there exist smooth functions $f_1, \ldots, f_{n-1}$ of one variable such that we have that

$$P_j(f_1(x_n), \ldots, f_{n-1}(x_n), x_n) = 0, j \in \{1, \ldots n-1\}. \tag{4.5}$$

Define the function φ that maps x_i to $z_i = x_i - f_i(x_n)$, for all $i \in \{1, \ldots, n-1\}$ and x_n to $z_n = x_n$. We can see that φ is a diffeomorphism and $z = (z_1, \ldots, z_n)$ is a local coordinate system around q. Hence, we can define the function $G_j(z) = P_j \circ \varphi^{-1}(z) = P_j(x)$ for all integers $1 \leqslant j \leqslant n-1$. Using Hadamard‘s Lemma [Dem00, Proposition 4.2.3] for the first $n-1$ variables of z, we can write $G_j(z) - G_j(0, \ldots, 0, z_n) = \sum_{i=1}^{n-1} z_i \cdot h_{ji}(z)$ for some smooth functions h_{ji}. Note that $\varphi^{-1}(z) = (z_1 + f_1(z_n), \ldots, z_{n-1} + f_{n-1}(z_n), z_n)$. Hence, $G_j(0, \ldots, 0, z_n) = P_j \circ \varphi^{-1}(0, \ldots, 0, z_n) = P_j(f_1(z_n), \ldots, f_{n-1}(z_n), z_n) = P_j(f_1(x_n), \ldots, f_{n-1}(x_n), x_n)$. The latter function is equal to zero by (4.5). Thus, $P_j(x) = G_j(z) = \sum_{i=1}^{n-1} z_i \cdot h_{ji}(z) = \sum_{i=1}^{n-1} (x_i - f_i(x_n)) \cdot H_{ji}(x)$, with $H_{ji}(x) = h_{ji} \circ \varphi(x)$.

Defining $M_0 = \Big(H_{ji}\Big)_{1 \leqslant j, i \leqslant n-1}$ we get:

$$\begin{pmatrix} P_1 \\ \cdots \\ \cdots \\ P_{n-1} \end{pmatrix} = M_0 \cdot \begin{pmatrix} x_1 - f_1(x_n) \\ \cdots \\ \cdots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix}.$$

Notice that M_0 evaluated at q is the invertible matrix $M_n(q)$. Hence, by continuity of the determinant function, there is a neighbourhood of q in which M_0 is invertible. Thus, writing M as the inverse of M_0 we get:

$$Q_0 = \begin{pmatrix} x_1 - f_1(x_n) \\ \cdots \\ \cdots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ \cdots \\ \cdots \\ P_{n-1} \end{pmatrix}. \tag{4.6}$$

To prove that $\min\{\operatorname{ord}(f_1(x_n)), \operatorname{ord}(f_2(x_n))\} > 1$, we take the Jacobian matrices of both sides of (4.6) and we evaluate them at $q = 0$. We get the equation $J_{Q_0}(q) = M(q) \cdot J_P(q)$. By invertibility of $M(q)$ we deduce that the k-th minors (obtained by removing the k-th column) of $J_{Q_0}(q)$ and $J_P(q)$ have the same rank. Computing $J_{Q_0}(q)$ and considering the fact that $\det(M_1(q)) = \det(M_2(q)) = 0$ implies that $f_1'(0) = f_2'(0) = 0$, we thus have that $\min\{\operatorname{ord}(f_1(x_n)), \operatorname{ord}(f_2(x_n))\}$ is at least two. $\square$

Lemma 46. *Preserving the notation and the assumptions in Lemma 45, the multiplicity m of the system $S = \{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$ at q is equal to $d = \min\{\operatorname{ord}(f_1(x_n)), \operatorname{ord}(f_2(x_n))\}$.*

Proof. First, we start with the case $m < \infty$. By Proposition 4, we can assume without loss of generality, that $f_1, \dots, f_{n-1}$ are polynomials. Following the notation in Definition 2, let $\mathbb{R}[x]$ (resp. $\mathbb{R}[x_n]$) be the ring of polynomials with n variables (resp. one variable) and $\mathbb{R}[x]_q$ (resp. $\mathbb{R}[x_n]_0$) be its localization at q (resp. $0 \in \mathbb{R}$). Also, define I_S to be the ideal generated by the polynomials of S in $\mathbb{R}[x]_q$ (as I_G is defined in Definition 2), i.e., $I_S = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots x_{n-1} - f_{n-1}(x_n), x_1, x_2\rangle = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots x_{n-1} - f_{n-1}(x_n), f_1(x_n), f_2(x_n)\rangle$. If $f_1(x_n) = f_2(x_n) = 0$, then the ideal I_S is of dimension one, hence, S has an infinite number of solutions which contradicts the assumption $m < \infty$. Thus, $d < \infty$ which means that there exist $h_1, h_2 \in \mathbb{R}[x_n]_0$ such that $h_1(x_n)f_1(x_n) + h_2(x_n)f_2(x_n) = x_n^d$. Thus, $I_S = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_n^d\rangle$. Note that the set $\{x_1 - f_1(x_n), x_2 - f_2(x_n), \dots x_{n-1} - f_{n-1}(x_n), x_n^d\}$ is a Gröbner basis of I_S with respect to Local Lexicographical ordering $x_1 > \cdots > x_n$. Hence, By [CLO05, Theorem 4.4.3] we have $\dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x]_q}{LT(I_S)}) = \dim(\frac{\mathbb{R}[x]_q}{\langle x_1, x_2, \dots x_{n-1}, x_n^d\rangle})$, where $LT(I_S)$ is the ideal generated by the leading terms of I_S. Consequently, $m = \dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x_n]_0}{\langle x_n^d\rangle}) = d$.

Second, assume that $m = \infty$. We prove that $d = \infty$, that is, $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$ for any positive integer k. Preserving the notation in Definition 3, consider the dual space $D_q^k[S]$. We are going to show that for any positive integer k and any element $c \in D_q^k[S] \setminus D_q^{k-1}[S]$ (which always exists since $m = \infty$), the coefficient $c_{x_n^k}$ corresponding to $\frac{\partial^k}{\partial x_n^k}$, for c, is non-zero. We consequentially show that $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$. We prove the previous statements by induction on k.

For $k = 1$, since $q \in \mathfrak{L}_{\mathrm{c}}$, we already showed in the proof of Lemma 8 that a non-trivial element $c = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$ is in $D_q^1[S] \setminus D_q^0[S]$ if and only if $v = (v_1, \dots, v_n)$ is in $T_q\mathfrak{C}$. On the other hand, $T_q\mathfrak{C}$ is generated by the vector $(f_1'(0), \dots f_{n-1}'(0), 1)$, thus $c_{x_n^1} = v_n \neq 0$. The function $f_1(x_n)$ is in the set of functions generated by S thus $0 = c \cdot (f_1(x_n)) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}_q \cdot (f_1(x_n)) = c_{x_n^1} \frac{\partial f_1}{\partial x_n}(0)$, and thus $\frac{\partial f_1}{\partial x_n}(0) = 0$. Thus, the induction hypothesis holds for $k = 1$.

Define $c' = \phi_n(c)$ and consider two cases:

(a) $c' \in D_q^{k-1}[S] \setminus D_q^{k-2}[S]$: By the induction hypothesis, the coefficient $c'_{x_n^{k-1}}$ corresponding to $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$ for c' is non-zero and $\frac{\partial^{k'} f_1}{\partial x_n^{k'}}(0) = \frac{\partial^{k'} f_2}{\partial x_n^{k'}}(0) = 0$, for all $k' < k$. Notice that by the definition of ϕ_n, we have $c_{x_n^k} = c'_{x_n^{k-1}} \neq 0$. Hence, $0 = c \cdot f_1(x_n) = \sum_{i=1}^{k} c_{x_n^i} \frac{\partial^i f_1}{\partial x_n^i}(0) = c_{x_n^k} \frac{\partial^k f_1}{\partial x_n^k}(0)$. Hence, $\frac{\partial^k f_1}{\partial x_n^k}(0) = 0$. Similarly, we prove that $\frac{\partial^k f_2}{\partial x_n^k}(0) = 0$. Thus in Case (a), the lemma is proved.

(b) $c' \in D_q^{k-2}[S]$: Since $c \in D_q^k[S] \setminus D_q^{k-1}[S]$, there exists $j \in \{1, \dots, n-1\}$ such that the element $c'' = \phi_j(c)$ is in $D_q^{k-1}[S] \setminus D_q^{k-2}[S]$. By the induction hypothesis, the coefficient $c''_{x_n^{k-1}}$ corresponding to $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$ for c'', is non-zero. On the other hand, $c_{x_j x_n^{k-1}} = c''_{x_n^{k-1}} \neq 0$. Hence, since $\phi_n(c_{x_j x_n^{k-1}} \frac{\partial^k}{\partial x_j \partial x_n^{k-1}}) \in D_q^{k-1}[S] \setminus D_q^{k-2}[S]$, then so is $\phi_n(c) = c'$ which contradicts the assumption. Thus, Case (b) is impossible.

$\square$

With the additional Assumptions $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, one can give a more precise form of f_1 and f_2 in Equation (4.4).

Lemma 47. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let $q \in \mathfrak{L}_{\mathrm{c}}$ such that Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ in $\overline{B}$, then there exist an invertible matrix $\widetilde{M}$ of size $(n-1) \times (n-1)$ of smooth functions in a neighbourhood of q, a smooth diffeomorphism φ defined in an open subset of $\mathbb{R}^n$, with $z = (z_1, \dots, z_n) = \varphi^{-1}(x)$ and smooth functions $f_3, \dots, f_{n-1}, g$ defined in a neighbourhood of $0 \in \mathbb{R}$, such that*

$$Q = \begin{pmatrix} z_1 - z_n \cdot g(z_n^2) \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \cdots \\ \cdots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} = \widetilde{M} \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \cdots \\ \cdots \\ P_{n-1} \end{pmatrix} \circ \varphi, \tag{4.7}$$

on a neighbourhood of q. Moreover, either $\operatorname{ord}(g(z_n)) = \infty$ or there exists an integer $k > 0$ with $g(z_n) = z_n^k$.

Proof. Step 1: Equation (4.6) implies that Q_0 and P define the same curve $\mathfrak{C}$ in a neighbourhood of q and that the function Q_0 satisfies the same assumptions as P around q. By Lemma 46, $d = \min\{\operatorname{ord}(f_1(x_n)), \operatorname{ord}(f_2(x_n))\}$ is the multiplicity of the system $\{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = 0, x_2 = 0\}$ at q. By Assumption $\mathcal{A}_3$, we have that $d = 2$.

Without loss of generality, assume that $\operatorname{ord}(f_2(x_n)) = 2$ and $\frac{\partial^2 f_2}{\partial x_n^2}(0) = 2$. Hence, there is a smooth function v such that $f_2(x_n) = x_n^2(1 + x_n \cdot v(x_n))$. Now, consider the diffeomorphism ϕ_n that sends x_n to $z_n = x_n\sqrt{1 + x_n \cdot v(x_n)}$. We have that $x_2 - f_2(x_n) = x_2 - z_n^2$. Define $\tilde{f}_1(z_n) = f_1(\phi_n^{-1}(z_n))$ and $\tilde{f}_2(z_n) = f_2(\phi_n^{-1}(z_n)) = z_n^2$. Since $\operatorname{ord}(\tilde{f}_1(z_n)) = \operatorname{ord}(f_1(x_n)) \geqslant d = 2$, there exists a smooth function h such that $\tilde{f}_1(z_n) = z_n^2 h(z_n)$. Write $\tilde{f}_1(z_n) = z_n^2[\frac{h(z_n)+h(-z_n)}{2} + \frac{h(z_n)-h(-z_n)}{2}]$. Since $\frac{h(z_n)+h(-z_n)}{2}$ (resp. $\frac{h(z_n)+h(-z_n)}{2}$) is even (resp. odd), then by Theorem 16 there exists a smooth function ξ_1 (resp. ξ_2) such that $\frac{h(z_n)+h(-z_n)}{2} = \xi_1(z_n^2)$ (resp. $\frac{h(z_n)-h(-z_n)}{2} = z_n \xi_2(z_n^2)$). Thus, $\tilde{f}_1(z_n) = z_n^2(\xi_1(z_n^2) + z_n \xi_2(z_n^2))$. Notice that $\xi_2(x_n^2)$ cannot be the zero function, otherwise $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$ and $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$ for all small enough $\epsilon > 0$, which contradicts Assumption $\mathcal{A}_4$.

Step 2: We have two cases:

Case 1: $\operatorname{ord}(\xi_2(z_n)) = \infty$, then define the diffeomorphism ϕ which sends x_1 to $z_1 = x_1 - x_2\xi_1(x_2)$, x_i to $z_i = x_i$ for all integers $i \in \{2, \dots, n-1\}$ and x_n to $z_n = x_n\sqrt{1 + x_n \cdot v(x_n)}$. Taking $g(z_n) = z_n\xi_2(z_n)$ and $\varphi = \phi^{-1}$ we prove the claim for the first case.

Case 2: $\operatorname{ord}(\xi_2(z_n)) = k < \infty$, that is, $\xi_2(z_n) = z_n^k u(z_n)$, for some smooth function u, with $u(0) \neq 0$ and an integer $k \geqslant 0$. Hence, we can write $x_1 - \tilde{f}_1(z_n) = x_1 - z_n^2\xi_1(z_n^2) - z_n^{2k+3}u(z_n^2) = x_1 - x_2\xi_1(x_2) - z_n^{2k+3}u(x_2)$.

So, defining the diffeomorphism ϕ which sends x_i to $z_i = x_i$ for all integers $i \in \{2, \dots, n-1\}$, x_n to $z_n = x_n\sqrt{1 + x_n \cdot v(x_n)}$ and x_1 to $z_1 = (x_1 - x_2\xi_1(x_2))u^{-1}(x_2)$ (which means that $x_1 - f_1(x_n) = u(x_2)[z_1 - z_n^{2k+3}]$), we get that:

$$\begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1\times(n-2)} \\ 0_{(n-2)\times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi, \tag{4.8}$$

for a small enough neighbourhood of q, where I_{n-2} is the identity matrix of size $n-2$. Comparing with (4.4), we get:

$$M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1\times(n-2)} \\ 0_{(n-2)\times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi. \tag{4.9}$$

Hence, taking $\widetilde{M} = \begin{pmatrix} u(x_2) & 0_{1\times(n-2)} \\ 0_{(n-2)\times 1} & I_{n-2} \end{pmatrix}^{-1} \cdot M$ and $\varphi = \phi^{-1}$ we recover (4.7). $\square$

Following the conclusion of Lemma 47, the reader may wonder whether the projection of q in $\pi_{\mathfrak{C}}$ is always singular. This is clear when $g(x_n) = x_n^k$ for $0 < k < \infty$ since this implies $z_1^2 - z_2^{k+1} = 0$ and thus $\pi_{\mathfrak{C}}(q)$ is a singularity of the type A_{2k}. We next prove that the projection is also singular if $\operatorname{ord}(g(z_n)) = \infty$.

Lemma 48. *Preserving the notation and the assumptions in Lemma 47, consider the function g defined in* (4.7)*, if $\operatorname{ord}(g(z_n)) = \infty$, then $\pi_{\mathfrak{C}}(q)$ is singular in $\pi_{\mathfrak{C}}(\mathfrak{C})$.*

Proof. Since $\operatorname{ord}(g(z_n)) = \infty$, then **Case 1** in the proof of Lemma 47 holds. Moreover, we saw in the same proof that $\xi_2(z_n^2)$ (restricted to an open neighbourhood of $0 \in \mathbb{R}$) cannot be the zero function. This implies that neither is the function $g(z_n^2) = z_n^2\xi(z_n^2)$, i.e., $g(z_n^2)$, restricted to an open neighbourhood of $0 \in \mathbb{R}$, is not the zero function. Assume for the sake of contradiction that $\pi_{\mathfrak{C}}(q)$ is smooth in $\pi_{\mathfrak{C}}(\mathfrak{C})$, then using the implicit function theorem, there exists a C^∞-function defined in a neighbourhood of 0 in $\mathbb{R}$, with $f(0) = 0$ such that for a small neighbourhood of $\pi_{\mathfrak{C}}(q)$ in $\mathbb{R}^2$, one of the following cases is satisfied:

(a) $f(z_1) = z_2 \iff (z_1, z_2) \in \pi_{\mathfrak{C}}(\mathfrak{C})$. Then, by (4.7), we have $f(z_n g(z_n^2)) = z_n^2$. Taking the second derivative of both sides with respect to z_n and then evaluating at 0 (recall that $\mathrm{ord}(g(z_n)) = \infty$), we get the contradiction $0 = 2$.

(b) $f(z_2) = z_1 \iff (z_1, z_2) \in \pi_{\mathfrak{C}}(\mathfrak{C})$. Then $f(z_n^2) = z_n g(z_n^2)$. The function $z_n g(z_n^2)$ is an odd function but not the zero function, and on the other hand $f(z_2)$ is an even function, which leads to a contradiction.

Thus, in both cases we have a contradiction, that is, f does not exist and $\pi_{\mathfrak{C}}(q)$ cannot be smooth in $\pi_{\mathfrak{C}}(\mathfrak{C})$. □

Returning to (4.7), notice that φ is defined in such a way that it preserves the singularity class of $\pi_{\mathfrak{C}}(\mathfrak{C})$ at the point $\pi_{\mathfrak{C}}(q)$. In other words, if C is the plane projection of the curve defined by the Q then $(\pi_{\mathfrak{C}}(\mathfrak{C}), 0)$ and $(C, 0)$ are equivalent. As a corollary of Lemmas 21, 22, 47 and 48, the points of $\mathfrak{C}$ in $\mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$ are projected to the singular points of $\pi_{\mathfrak{C}}(\mathfrak{C})$.

Corollary 49. *If P satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, then a point $q \in \mathfrak{C}$ projects to a singular point in $\pi_{\mathfrak{C}}(\mathfrak{C})$ if and only if $q \in \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$.*

Proof. If $q \in \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$, then by Lemmas 22, 47, and 48, $\pi_{\mathfrak{C}}(q)$ is singular in $\pi_{\mathfrak{C}}(\mathfrak{C})$. If $q \notin \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$, then by Lemma 21, $\pi_{\mathfrak{C}}(q)$ is smooth in $\pi_{\mathfrak{C}}(\mathfrak{C})$. □

Finally, we prove that the solutions of the ball system project to the singular points of $\pi_{\mathfrak{C}}(\mathfrak{C})$.

Proof of Theorem 36: By Corollary 49, if (x_1, x_2) is singular in $\pi_{\mathfrak{C}}(\mathfrak{C})$, then there exists a point $q_1 \in \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$, with $\pi_{\mathfrak{C}}(q_1) = (x_1, x_2)$. If $q_1 \in \mathfrak{L}_{\mathrm{c}}$, let $q_2 = q_1$ and otherwise let q_2 be the unique (by Assumption $\mathcal{A}_3$) point in $\mathfrak{L}_{\mathrm{n}}$, distinct from q_1, that projects onto (x_1, x_2), i.e. $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2) = (x_1, x_2)$. Hence, (q_1, q_2) is in $\widehat{\mathfrak{L}}$. Since Ω_P is surjective (Lemma 40), there exists $X = (x_1, x_2, y, r, t) \in Sol_{\mathrm{Ball}(P)}$ with $\Omega_P(X) = (q_1, q_2)$.

On the other hand, if X is a solution of $\mathrm{Ball}(P)$, then by Lemma 38 the pair $(q_1, q_2) = \Omega_P(X)$ is in $\widehat{\mathfrak{L}}$. Hence, $q_1 = (x_1, x_2, y + r\sqrt{t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ is in $\mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$. Hence, by Corollary 49 the point (x_1, x_2) is singular in $\pi_{\mathfrak{C}}(\mathfrak{C})$. □

4.3. *Regularity of the ball system*

In this section, our goal is to prove Theorem 51 determining necessary and sufficient conditions for $\mathrm{Ball}(P)$ to be regular. We first recall the definition of a regular system.

Definition 50. *For some integer $m \leqslant n$, let $F = (f_1, \ldots, f_m)$ be a vector of smooth real-valued functions that are defined in $\mathbb{R}^n$ and let $a \in \mathbb{R}^n$ be a solution of the system $\{F = 0\}$. We say that the latter system is regular at $a \in \mathbb{R}^n$ if the rank of its Jacobian matrix, evaluated at a, equals to m. We call $\{F = 0\}$ regular if it is regular at all of its solutions.*

Theorem 51. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ that satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$,$\mathcal{A}_3$ and $\mathcal{A}_4$, then P satisfies Assumption $\mathcal{A}_5$ if and only if $\mathrm{Ball}(P)$ is regular in B_{Ball}.*

In order to prove Theorem 51, we are going to show that the Jacobian matrices of $\mathrm{Ball}(P)$ and $\mathrm{Ball}(Q)$ evaluated at X have the same rank, where Q is defined in Equation (4.7). Recall that Equation (4.7) implies that P and Q define the same curve around q. Notice also that if $X = (q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ is in $\Omega_P^{-1}((q, q))$, then $X \in \Omega_Q^{-1}((q, q))$.

Lemma 52. *Let P and Q be as defined in* (4.7)*. Under Assumption $\mathcal{A}_1$, let $(q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ be a solution of the system* $\mathrm{Ball}(P)$ *in B_{Ball}, then* $\mathrm{Ball}(P)$ *is regular at $(q, r, 0)$ if and only if* $\mathrm{Ball}(Q)$ *is regular at the point* $(0, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ *(recall that for simplicity, we assume in Lemma 47 that $q = 0 \in \mathbb{R}^n$).*

Proof. Let us write $X = (q, r, 0)$. We are going to prove that the Jacobian matrices of $\mathrm{Ball}(P)$ and $\mathrm{Ball}(Q)$ evaluated at X have the same rank. By Remark 41 we have that $\Omega_P(X) = (q, q) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$ (see Definitions 39 and 37), and hence, $q \in \mathfrak{L}_{\mathrm{c}}$. By Lemma 38 we have that $(0, 0, r) \in T_q\mathfrak{C}$. We prove the claim in three steps:

Step 1: Let $\widetilde{M} = (f_{ij})_{1 \leqslant i,j \leqslant n-1}$ be as defined in the Equality (4.7). We define $S \cdot \widetilde{M}$ (resp. $D \cdot \widetilde{M}$) to be the matrix $(S \cdot f_{ij})_{1 \leqslant i,j \leqslant n-1}$ (resp. $(D \cdot f_{ij})_{1 \leqslant i,j \leqslant n-1}$). Using the identity $\frac{1}{2}(ab + cd) = \frac{1}{4}(a + c)(b + d) + \frac{1}{4}(a - c)(b - d)$, one deduces the properties for any $f, g \in C^\infty(\mathbb{R}^n, \mathbb{R})$:

$$S \cdot fg = (S \cdot f)(S \cdot g) + t(D \cdot f)(D \cdot g) \tag{4.10}$$

$$D \cdot fg = (D \cdot f)(S \cdot g) + (S \cdot f)(D \cdot g) \tag{4.11}$$

These identities applied to Equation (4.7) yield

$$\begin{pmatrix} S \cdot Q_1 \\ \dots \\ \dots \\ S \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \dots \\ \dots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \dots \\ \dots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

and

$$\begin{pmatrix} D \cdot Q_1 \\ \dots \\ \dots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \dots \\ \dots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \dots \\ \dots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

Combining the last two equalities:

$$\begin{pmatrix} S\cdot Q_1 \\ \dots \\ \dots \\ S\cdot Q_{n-1} \\ D\cdot Q_1 \\ \dots \\ \dots \\ D\cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S\cdot \widetilde{M} & tD\cdot \widetilde{M} \\ D\cdot \widetilde{M} & S\cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S\cdot (P_1\circ\varphi) \\ \dots \\ \dots \\ S\cdot (P_{n-1}\circ\varphi) \\ D\cdot (P_1\circ\varphi) \\ \dots \\ \dots \\ D\cdot (P_{n-1}\circ\varphi) \end{pmatrix} \tag{4.12}$$

Notice that $\begin{pmatrix} S\cdot \widetilde{M} & tD\cdot \widetilde{M} \\ D\cdot \widetilde{M} & S\cdot \widetilde{M} \end{pmatrix}_X = \begin{pmatrix} \widetilde{M}(q) & 0 \\ D\cdot \widetilde{M}(X) & \widetilde{M}(q) \end{pmatrix}$ (recall that on our case we have $S\cdot \widetilde{M}(X) = \widetilde{M}(q)$) and that the latter matrix has an inverse (recall that, by Lemma 47, $\widetilde{M}(q)$ is an invertible matrix of size $n-1$), namely, $\begin{pmatrix} \widetilde{M}(q)^{-1} & 0 \\ -\widetilde{M}(q)^{-1}\cdot (D\cdot\widetilde{M})(X)\cdot \widetilde{M}(q)^{-1} & \widetilde{M}(q)^{-1} \end{pmatrix}$ which implies (by continuity of the determinant function) that $\begin{pmatrix} S\cdot \widetilde{M} & tD\cdot \widetilde{M} \\ D\cdot \widetilde{M} & S\cdot \widetilde{M} \end{pmatrix}$ is invertible in a neighbourhood of X.

Step 2: Writing $y = (y_3, \dots, y_n)$ and $r = (r_3, \dots, r_n)$, consider the diffeomorphism φ defined in Lemma 47 and define the smooth function ψ over an open subset of $\mathbb{R}^{2n-1}$ containing X which maps the point (x_1, x_2, y, r, t) to $(\varphi_1, \varphi_2, S\cdot\varphi_3, \dots, S\cdot\varphi_n, D\cdot\varphi_3, \dots, D\cdot\varphi_n, t)$. Notice that we have:

$$S\cdot (P_j\circ\varphi) = (S\cdot P)\circ\psi \text{ and } D\cdot (P_j\circ\varphi) = (D\cdot P)\circ\psi, \text{ for } 1\leqslant j\leqslant n-1, \tag{4.13}$$

since $\varphi_i(x_1, x_2, y\pm r\sqrt{t}) = \psi_i \pm \psi_{n+i-2}\sqrt{\psi_{2n-1}}$ for all $i\in\{3,\dots,n\}$. In fact, using the last two equalities we can also see that ψ^{-1} exists and is smooth. Thus, ψ is a diffeomorphism.

Step 3: Now, comparing (4.12) with (4.13) we get:

$$SD\cdot Q := \begin{pmatrix} S\cdot Q_1 \\ \dots \\ \dots \\ S\cdot Q_{n-1} \\ D\cdot Q_1 \\ \dots \\ \dots \\ D\cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S\cdot \widetilde{M} & tD\cdot \widetilde{M} \\ D\cdot \widetilde{M} & S\cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S\cdot P_1 \\ \dots \\ \dots \\ S\cdot P_{n-1} \\ D\cdot P_1 \\ \dots \\ \dots \\ D\cdot P_{n-1} \end{pmatrix} \circ\psi.$$

Consider the vector $SD\cdot P = (S\cdot P_1, \dots, S\cdot P_{n-1}, D\cdot P_1, \dots, D\cdot P_{n-1})^T$ and let $J_{SD\cdot P}$, $J_{SD\cdot Q}$ and J_ψ be the

Jacobian matrices of $SD \cdot P, SD \cdot Q$ and ψ respectively. Taking the Jacobian matrix of both sides of the last equality:

$$J_{SD\cdot Q} = \begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} \end{pmatrix} \cdot J_{SD\cdot P} \cdot J_{\psi} + \text{Jacobian}(\begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} \end{pmatrix}) \cdot \begin{pmatrix} S\cdot P_1 \\ \dots \\ \dots \\ S\cdot P_{n-1} \\ D\cdot P_1 \\ \dots \\ \dots \\ D\cdot P_{n-1} \end{pmatrix} \circ \psi.$$

Evaluating the last equality at $X = (0, r, 0)$ and using the fact that $\psi(X) = \psi(0, r, 0) = (0, r, 0) = X$, we note that the second term of the right-hand side is zero. One thus has:

$$J_{SD\cdot Q}(X) = \begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} \end{pmatrix}_X \cdot J_{SD\cdot P}(X) \cdot J_{\psi}(X). \tag{4.14}$$

Computing $J_{\psi}(X)$, we get $J_{\psi}(X) = \left(\begin{array}{c:cc} \frac{\partial\varphi_1}{\partial z_1}(0) & \frac{\partial\varphi_1}{\partial z_2}(0) & 0_{1\times(2n-3)} \\ \hdashline 0_{(2n-2)\times 1} & \multicolumn{2}{c}{I_{2n-2}} \end{array}\right)$, with $\frac{\partial\varphi_1}{\partial z_1}(0) \neq 0$ according to the formula in Lemma 47.

Hence by Equation (4.14), it is straightforward to check that:

$$J_{\text{Ball}(Q)} = \begin{pmatrix} J_{SD\cdot Q}(X) \\ 2X \end{pmatrix} = \begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} & 0 \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} & 0 \\ 0_{1\times(n-1)} & 0_{1\times(n-1)} & 1 \end{pmatrix}_X \cdot \begin{pmatrix} J_{SD\cdot P}(X) \\ 2X \end{pmatrix} \cdot J_{\psi}(X) = \begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} & 0 \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} & 0 \\ 0_{1\times(n-1)} & 0_{1\times(n-1)} & 1 \end{pmatrix}_X \cdot J_{\text{Ball}(P)}(X) \cdot J_{\psi}(X).$$

Recalling that $J_{\psi}(X)$ and $\begin{pmatrix} S\cdot\widetilde{M} & tD\cdot\widetilde{M} & 0 \\ D\cdot\widetilde{M} & S\cdot\widetilde{M} & 0 \\ 0_{1\times(n-1)} & 0_{1\times(n-1)} & 1 \end{pmatrix}_X$ are invertible matrices, the proof of the lemma follows. □

Now, we are ready to prove Theorem 51 which characterizes the regularity of the solutions of $\text{Ball}(P)$ under generic assumptions. We split the proof in two Lemmas 54 and 55. Before that, we introduce a new assumption that helps to simplify the proof.

Definition 53. *Let $(q_1, q_2) \in \widehat{\mathfrak{L}}$. We say that (q_1, q_2) satisfies Assumption $\mathcal{A}_5'$ if q_1 and q_2 are isolated in $\mathfrak{L}_{\text{n}} \cup \mathfrak{L}_{\text{c}}$ and any of the following conditions is satisfied:*

(a) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\text{n}}$, then the plane projections of the tangent lines of q_1 and q_2 to $\mathfrak{C}$ are linearly independent.

(b) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_{\text{c}}$, then the plane projection of a small enough neighbourhood of q_1 in $\mathfrak{C}$ is an ordinary cusp at $\pi_{\mathfrak{C}}(q_1)$ and the multiplicity of the system $\{P(x) = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$ at q_1 is two.

Assumption $\mathcal{A}_5'$ can be seen as a "local version" of Assumption $\mathcal{A}_5$. We are going to prove that if Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are satisfied, then Assumption $\mathcal{A}_5$ is equivalent to the condition that Assumption $\mathcal{A}_5'$ is satisfied for all $\widehat{\mathfrak{L}}$.

The main reason behind introducing Assumption $\mathcal{A}_5'$, is that we are going to prove in Lemma 54 that, under Assumption $\mathcal{A}_1$, a pair $(q_1, q_2) \in \widehat{\mathfrak{L}}$ satisfies Assumption $\mathcal{A}_5'$ if and only if every X in $\Omega_P^{-1}((q_1, q_2))$ is a regular solution of $\mathrm{Ball}(P)$, whereas Assumption $\mathcal{A}_5$ is, in general, not sufficient for the regularity of the solutions of $\mathrm{Ball}(P)$. For example, take $n = 3$ and $P = (x_1 - x_3^6, x_2 - x_3^9)$. We can see that P satisfies Assumption $\mathcal{A}_1$, the set $\mathfrak{L}_{\mathrm{c}}$ consists of a unique point $q = (0, 0, 0)$ and the set $\mathfrak{L}_{\mathrm{n}}$ is empty. The plane projection of $\mathfrak{C}$ is the curve given by the equation $x_1^3 - x_2^2 = 0$. Hence, Assumption $\mathcal{A}_5$ is satisfied. However, the multiplicity of the system $\{P(x_1, x_2, x_3) = 0 \in \mathbb{R}^2, x_1 = x_2 = 0\}$ at the point q equals to 6 (Lemma 46). Hence, Assumption $\mathcal{A}_5'$ is not satisfied and one can also check that $\mathrm{Ball}(P)$ is not regular.

Lemma 54. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ that satisfies Assumption $\mathcal{A}_1$. Let X be a solution of* $\mathrm{Ball}(P)$ *and $(q_1, q_2) = \Omega_P(X)$ (Definition 39), then X is a regular solution of* $\mathrm{Ball}(P)$ *if and only if (q_1, q_2) satisfies Assumption $\mathcal{A}_5'$.*

Proof. Let $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ be a solution of $\mathrm{Ball}(P)$. We consider two cases:

Case (a). $t \neq 0$, i.e., $q_1 \neq q_2$.

It is easy to see that $\frac{\partial(S\cdot P_i)}{\partial x_j}, \frac{\partial(D\cdot P_i)}{\partial x_j}, \frac{\partial(S\cdot P_i)}{\partial r_k}, \frac{\partial(D\cdot P_i)}{\partial r_k}, \frac{\partial(S\cdot P_i)}{\partial t}, \frac{\partial(D\cdot P_i)}{\partial t}$ are respectively equal to: $S \cdot \frac{\partial(P_i)}{\partial x_j}, D \cdot \frac{\partial(P_i)}{\partial x_j}, t \cdot D \cdot \frac{\partial(P_i)}{\partial x_k}, S \cdot \frac{\partial(P_i)}{\partial x_k}, \frac{1}{2} \sum\limits_{m=3}^{n} D \cdot (\frac{\partial P_i}{\partial x_m}) \cdot r_m, \frac{1}{2t}[\sum\limits_{m=3}^{n} S \cdot (\frac{\partial P_i}{\partial x_m}) \cdot r_m - D \cdot P_i]$. Hence, by computing the Jacobian matrix of the $\mathrm{Ball}(P)$ we get the matrix:

$$\begin{pmatrix}
S \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & S \cdot \frac{\partial P_1}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2}\sum\limits_{m=3}^{n} D \cdot (\frac{\partial P_1}{\partial x_m}) \cdot r_m \\
 & \dots & \dots & \dots & \dots & \dots \\
 & \dots & \dots & \dots & \dots & \dots \\
S \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2}\sum\limits_{m=3}^{n} D \cdot (\frac{\partial P_{n-1}}{\partial x_m}) \cdot r_m \\
D \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & D \cdot \frac{\partial(P_1)}{\partial x_n} & S \cdot \frac{\partial(P_1)}{\partial x_3} \dots & S \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2t}[\sum\limits_{m=3}^{n} S \cdot (\frac{\partial P_1}{\partial x_m}) \cdot r_m - D \cdot P_1] \\
\dots & \dots & \dots & \dots & \dots & \\
\dots & \dots & \dots & \dots & \dots & \\
D \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & S \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2t}[\sum\limits_{m=3}^{n} S \cdot (\frac{\partial P_{n-1}}{\partial x_m}) \cdot r_m - D \cdot P_{n-1}] \\
0 & \dots & 0 & 2r_3 \dots & 2r_n & 0
\end{pmatrix}. \tag{4.15}$$

We denote by C_i (resp. L_i) the i-th column (resp. line) of the latter matrix. Replace the last column C_{2n-1} with $\sum\limits_{m=1}^{n-2} \frac{r_{m+2}}{2t} C_{n+m} + C_{2n-1}$, also for all integers $1 \leqslant k \leqslant n-1$ we replace the line L_k with $L_k + \sqrt{t} \cdot L_{k+n-1}$ and then the line L_{k+n-1} with $L_k - 2\sqrt{t}L_{k+n-1}$. The resulting matrix is:

$$\begin{pmatrix}
\frac{\partial(P_1)}{\partial x_1}(q_1) & \dots \frac{\partial P_1}{\partial x_n}(q_1) & \sqrt(t) \cdot \frac{\partial(P_1)}{\partial x_3}(q_1) & \dots & \sqrt(t)\frac{\partial(P_1)}{\partial x_n}(q_1) & 0 \\
 & \dots & \dots & \dots & \dots & \dots \\
 & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial(P_{n-1})}{\partial x_1}(q_1) & \dots \frac{\partial P_{n-1}}{\partial x_n}(q_1) & \sqrt(t) \cdot \frac{\partial(P_{n-1})}{\partial x_3}(q_1) & \dots & \sqrt(t)\frac{\partial(P_{n-1})}{\partial x_n}(q_1)) & 0 \\
\frac{\partial(P_1)}{\partial x_1}(q_2) & \dots \frac{\partial(P_1)}{\partial x_n}(q_2) & -\sqrt(t)\frac{\partial(P_1)}{\partial x_3}(q_2) & \dots & -\sqrt(t)\frac{\partial(P_1)}{\partial x_n}(q_2) & 0 \\
\dots & \dots & \dots & \dots & \dots & \\
 & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial(P_{n-1})}{\partial x_1}(q_2) & \dots \frac{\partial(P_{n-1})}{\partial x_n}(q_2) & -\sqrt(t)\frac{\partial(P_{n-1})}{\partial x_3}(q_2) & \dots & -\sqrt(t)\frac{\partial(P_{n-1})}{\partial x_n}(q_2)) & 0 \\
0 \dots & 0 & 2r_3 & \dots & 2r_n & \frac{1}{2t}
\end{pmatrix}.$$

The determinant of the latter matrix is zero if and only if the determinant of the following matrix is zero:

$M_0 = \begin{pmatrix} N_P(q_1) & M_P(q_1) & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & -M_P(q_2) \end{pmatrix}$, where $M_P(q_1), M_P(q_2)$ are the minors that are obtained respectively by removing the first two columns from $J_P(q_1), J_P(q_2)$ and $N_P(q_1), N_P(q_2)$ are the matrices formed by the first two columns of $J_P(q_1), J_P(q_2)$ respectively. By linear operations on M_0, we can see that M_0 has same rank as the matrix $M(q_1, q_2)$ (see Definition 23). Thus, X is regular for $\mathrm{Ball}(P)$ if and only if $M(q_1, q_2)$ is invertible. By

Lemma 24 we have that $M(q_1, q_2)$ is invertible if and only if none of q_1, q_2 is in $\mathfrak{L}_{\mathrm{c}}$ (and hence none of the plane projections of $T_{q_1}\mathfrak{C}, T_{q_2}\mathfrak{C}$ is trivial) and the plane projection of their tangent spaces are different. Equivalently, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_{\mathrm{n}}$ and satisfies Assumption $\mathcal{A}'_5$.

Case (b) $t = 0$**, i.e.,** $q_1 = q_2$**.**

Let us write $q = q_1$. We prove the claim in three steps:

Step 1: We first simplify P. Without loss of generality and by Lemma 45 we can assume that $q = 0$ and $P_1, \ldots, P_{n-1}$ are respectively equal to $x_1 - f_1(x_n), x_2 - f_2(x_n), \ldots, x_{n-1} - f_{n-1}(x_n)$ with the property that $\min\{\mathrm{ord}(f_1), \mathrm{ord}(f_2)\} \geqslant 2$. For all $i \in \{3, \ldots, n-1\}$, using Taylor's theorem, we can write $f_i(x_n) = \sum_{j=1}^{3} a_{i,j} x_n^j + x_n^4 h_i(x_n)$, for some $a_{i,j} \in \mathbb{R}$ and smooth functions $h_i(x_n)$. Since $\min\{\mathrm{ord}(f_1), \mathrm{ord}(f_2)\} \geqslant 2$, we can write $f_1(x_n) = \sum_{j=2}^{3} \alpha_j x_n^j + x_n^4 h_1(x_n)$ and $f_2(x_n) = \sum_{j=2}^{3} \beta_j x_n^j + x_n^4 h_2(x_n)$. Notice that

$$(f_1(x_n), f_2(x_n), f_3(x_n), \ldots, f_{n-1}(x_n), x_n)$$

is a local parametrization system of $\mathfrak{C}$ around q. Since $\dim(T_q\mathfrak{C}) = 1$ (Assumption $\mathcal{A}_1$), there exists $\lambda \in \mathbb{R}^*$ with $(a_{3,1}, \ldots, a_{n-1,1}, 1) = \lambda r$ (because the vectors $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ and $(0, 0, a_{1,3}, \ldots, a_{1,n-1}, 1)$ are in $T_q\mathfrak{C} \setminus \{0\}$). In particular, $r_n \neq 0$.

Step 2: Now, we compute $J_{\mathrm{Ball}(P)}(X)$ by first computing it for X_t, which is X but its last variable $t \neq 0$, and then taking the limit when t goes to 0. The operator S being linear, we can write $S(x_i - f_i(x_n)) = S(x_i - \sum_{j=1}^{3} a_{i,j} x_n^j) - S(x_n^4 h_i(x_n))$. On the other hand, using the identity (4.10) we deduce that $S(x_n^4 h_i(x_n)) = S(x_n^4) \cdot S(h_i(x_n)) + tD(x_n^4) \cdot D(h_i(x_n))$, for all $\in \{1, \ldots, n-1\}$. It is straightforward to see that $S(x_n^4) = r_n^4 t^2 + 6 r_n^2 t x_n^2 + x_n^4$ and $tD(x_n^4) = 4 r_n^3 x_n t^2 + 4 r_n x_n^3 t$ with $r = (r_3, \ldots, r_n)$. Hence, all of the first-order partial derivatives of $S(x_n^4 h_i(x_n))$, evaluated at X_t, converge to zero when t goes to 0. Hence, the evaluation of the partial derivatives of the functions $S(x_i - f_i(x_n))$ and $S(x_i - \sum_{j=1}^{3} a_{i,j} x_n^j)$, at X are equal. Using an analogical argument, we deduce that the evaluation of the partial derivatives of the functions $D(x_i - f_i(x_n))$ and $D(x_i - \sum_{j=1}^{3} a_{i,j} x_n^j)$, at X are also equal. Thus, $J_{\mathrm{Ball}(P)}(X_t)$ and $J_{\mathrm{Ball}(\overline{P})}(X_t)$ converge to the same limit $J_{\mathrm{Ball}(P)}(X)$, where $\overline{P}$ is the function obtained by truncating P beyond degree 3 with respect to the variable x_n.

Computing $J_{\mathrm{Ball}(P)}(X) = \lim_{t\to 0} J_{\mathrm{Ball}(\overline{P})}(X_t)$, we get:

$$\begin{pmatrix}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -\alpha_2 r_n^2 \\
0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & -\beta_2 r_n^2 \\
0 & 0 & \dots & \dots & -a_{3,1} & 0 & \dots & 0 & -a_{3,2} r_n^2 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & 0 & \dots & 1 & -a_{n-1,1} & 0 & \dots & 0 & -a_{n-1,2} r_n^2 \\
0 & 0 & \dots & \dots & -2\alpha_2 r_n & 0 & \dots & 0 & -\alpha_3 r_n^3 \\
0 & 0 & \dots & \dots & -2\beta_2 r_n & 0 & \dots & 0 & -\beta_3 r_n^3 \\
0 & 0 & \dots & \dots & -2a_{3,2} r_n & 1 & \dots & -a_{3,1} & -a_{3,3} r_n^3 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & 0 & \dots & \dots & -2a_{n-1,2} r_n & 0 & \dots 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\
0 & 0 & \dots & \dots & 0 & 2r_3 & \dots 2r_{n-1} & 2r_n & 0
\end{pmatrix}. \tag{4.16}$$

Hence, observing that the matrix is block diagonal, its determinant is zero if and only if the determinant of the following one is:

$$\begin{pmatrix}
-2\alpha_2 r_n & 0 & \dots & 0 & 0 & -\alpha_3 r_n^3 \\
-2\beta_2 r_n & 0 & \dots & 0 & 0 & -\beta_3 r_n^3 \\
-2a_{3,2} r_n & 1 & 0\dots & 0 & -a_{3,1} & -a_{3,3} r_n^3 \\
 & \dots & \dots & & & \\
-2a_{n-1,2} r_n & 0 & 0\dots & 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\
0 & 2r_3 & \dots & 2r_{n-1} & 2r_n & 0
\end{pmatrix}.$$

Shifting the columns of the last matrix we get:

$$\begin{pmatrix}
-\alpha_3 r_n^3 & -2\alpha_2 r_n & 0 & \dots & 0 & 0 \\
-\beta_3 r_n^3 & -2\beta_2 r_n & 0 & \dots & 0 & 0 \\
-a_{3,3} r_n^3 & -2a_{3,2} r_n & 1 & 0\dots & 0 & -a_{3,1} \\
 & \dots & \dots & & & \\
-a_{n-1,3} r_n^3 & -2a_{n-1,2} r_n & 0 & 0\dots & 1 & -a_{n-1,1} \\
0 & 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n
\end{pmatrix}.$$

To compute the determinant of the second block, we expand it about the last row. Hence, the determinant of the last matrix is zero if and only if $r_n(\alpha_2\beta_3 - \alpha_3\beta_2)(r_n + \sum_{i=3}^{n-1} a_{i,1} r_i) = 0$. Notice that, by Step 1, we have that $r_n \neq 0$ and the third factor $(r_n + \sum_{i=3}^{n-1} a_{i,1} r_i)$ is never zero since it is equal to λ. Thus, $J_{\mathrm{Ball}(P)}(X)$ is invertible iff $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$, equivalently, the matrix $A = \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}$ is invertible.

Step 3: We now show that the invertibility of A is equivalent to the condition that (q,q) satisfies Assumption $\mathcal{A}_5'$. First assume that A is invertible. It follows that either $\alpha_2 \neq 0$ or $\beta_2 \neq 0$ and this yields that the minimum of the orders of f_1 and f_2 is 2. By Lemma 46, the multiplicity of the system $\{P(x_1,x_2,y)=0 \in \mathbb{R}^{n-1}, (x_1,x_2)=\pi_{\mathfrak{C}}(q)\}$ at q is equal to 2, thus Assumption $\mathcal{A}_5'$ (b) is satisfied. Using the same notation as in the proof of Lemma 47, one can write $\tilde{f}_1(z_n) = z_n^2(\xi_1(z_n^2) + z_n\xi_2(z_n^2))$. Notice that $\xi_2(x_n^2)$ cannot be the zero function, otherwise $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$ and $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$ for all small enough $\epsilon > 0$, which means that X would be the limit of solutions X_ϵ of $\mathrm{Ball}(P)$ with $\Omega_P(X_\epsilon) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$. X would then be a non-isolated solution and thus a non-regular solution of $\mathrm{Ball}(P)$ which contradicts the assumption. We then have two cases as in Lemma 47. The first one is when $\mathrm{ord}(\xi_2(z_n)) = \infty$, that would imply that $\alpha_2 = \alpha_3 = 0$ and contradicts the invertibility of A. We then must satisfy the second case $\mathrm{ord}(\xi_2(z_n)) = k < \infty$ and, after a change of variables, the first equation of the system becomes equivalent to $z_1 - z_n^{2k+3} = 0$. The invertibility of A implies that $k=0$. The projection of the curve in the plane is thus locally parametrized by (z_n^3, z_n^2) and is an ordinary cusp, Assumption $\mathcal{A}_5'$ (a) is satisfied.

Second, assume that Assumption $\mathcal{A}_5'$ is satisfied. By Lemma 46 and Assumption $\mathcal{A}_5'$ (b), the minimum of the orders of f_1 and f_2 is 2. Using again the proof of Lemma 47, one can assume that $f_2(z_n) = z_n^2$ and $f_1(z_n) = z_n g(z_n^2)$ or $f_1(z_n) = z_n^{2k+3}$. By Assumption $\mathcal{A}_5'$ (a), the projection is an ordinary cusp and thus has a parametrization of the form (z_n^2, z_n^3), that is $f_1(z_n) = z_n^3$. This implies that A is equivalent to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and hence is invertible. $\square$

Lemma 55. *If Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, then Assumption $\mathcal{A}_5$ is satisfied if and only if Assumption $\mathcal{A}_5'$ is satisfied for all $(q_1,q_2) \in \widehat{\mathfrak{L}} \subset B \times B$.*

Proof. Assume that Assumption $\mathcal{A}_5$ is satisfied and $(q_1,q_2) \in \widehat{\mathfrak{L}}$. If $(q_1,q_2) \in \widehat{\mathfrak{L}}_{\mathrm{c}}$, then by Lemma 47 and Assumption $\mathcal{A}_5$ we must have that the plane projection of a small enough neighbourhood of q_1 in $\mathfrak{C}$ is an ordinary cusp at $\pi_{\mathfrak{C}}(q_1)$. By Assumption $\mathcal{A}_3$ and Lemma 8, the multiplicity of the mentioned system at $q_1 = q_2$ is two. Thus, (q_1,q_2) satisfies Assumption $\mathcal{A}_5'$. If $(q_1,q_2) \in \widehat{\mathfrak{L}}_{\mathrm{n}}$, then by Lemma 22 and Assumption $\mathcal{A}_5$, we have that $\pi_{\mathfrak{C}}(q_1)$ is a node in $\pi_{\mathfrak{C}}(\mathfrak{C})$. Thus, we have that $\pi_{\mathfrak{C}}(q_1)$ is a transverse intersection of two smooth branches of $\pi_{\mathfrak{C}}(\mathfrak{C})$. Those branches are the plane projections of two disjoint branches of $\mathfrak{C}$ each of which contains either q_1 or q_2. Hence, the plane projections of the tangent spaces of q_1 and q_2 to $\mathfrak{C}$ are linearly independent. Thus, (q_1,q_2) satisfies Assumption $\mathcal{A}_5'$.

Assume conversely that $\mathcal{A}_5'$ is satisfied for all $(q_1,q_2) \in \widehat{\mathfrak{L}}$. By Corollary 49, any singular point of $\pi_{\mathfrak{C}}(\mathfrak{C})$ is the plane projection of a point $q_1 \in \mathfrak{L}_{\mathrm{c}} \cup \mathfrak{L}_{\mathrm{n}}$. For some $q_2 \in \mathfrak{C}$, the pair (q_1,q_2) is in $\widehat{\mathfrak{L}}$ (which satisfies Assumption $\mathcal{A}_5'$). Hence, if (q_1,q_2) is in $\widehat{\mathfrak{L}}_{\mathrm{n}}$ (resp. in $\widehat{\mathfrak{L}}_{\mathrm{c}}$) the plane projection of q_1 is a node (resp. an ordinary cusp) by Lemma 22 (resp. Lemma 47). $\square$

Example 56. *Consider $n=4$ and let us represent $\mathbb{R}^4$ by the variables x,y,z,h. Let B the subset of $\mathbb{R}^4$ with $x,y,z \in [-1,4]$ and $h \in [-\frac{3\pi}{2} - 0.1, -\frac{pi}{2} + 0.1]$. Define over B the function $P_1 = x - \cos(h)(3+\sin^4(h)) + 3$, $P_2 = y - \sin^2(h)(3+\sin(2h))$, $z - h^2$ and $P = (P_1,P_2,P_3)$.*

We can see that $J_P(q)$ is full rank for all $q \in \mathfrak{C}$, thus Assumption $\mathcal{A}_1$ is satisfied by P. Moreover, the set $\mathfrak{L}_{\mathrm{n}}$ consists of the points corresponding to the values $h = \frac{-3\pi}{2}, \frac{-\pi}{2}$. Also, the set $\mathfrak{L}_{\mathrm{c}}$ has a unique point, namely, the

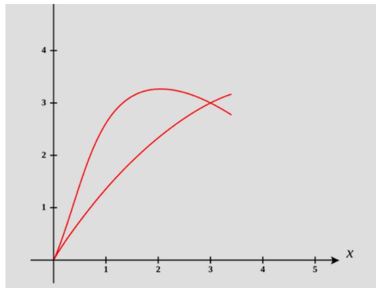

Figure 4: The plane projection of $\mathfrak{C}$.

point corresponding to the value $h = -\pi$. Hence, Assumptions $\mathcal{A}_2$ and $\mathcal{A}_4$ are satisfied. We can also check that Assumptions $\mathcal{A}_3$ and $\mathcal{A}_5$ are satisfied using the Taylor expansion of P_1, P_2 and P_3 centered at the points of $\mathfrak{L}_\mathrm{c}$ and $\mathfrak{L}_\mathrm{n}$ separately. Thus, P satisfies the assumptions in Section 2. The plane projection of $\mathfrak{C}$ is a singular curve (Figure 4) that has one node and one ordinary cusp.

Now, Computing $\mathrm{Ball}(P)$ *we get:*

$$\begin{cases} S \cdot P_1 = x - S \cdot [\cos(h)(3 + \sin^4(h))] + 3 = 0 \\ S \cdot P_2 = y - S \cdot [\sin^2(h)(3 + \sin(8h))] = 0 \\ S \cdot P_3 = z - h^2 - r_4^2 t = 0 \\ D \cdot P_1 = -D \cdot [\cos(h)(3 + \sin^4(h))] = 0 \\ D \cdot P_2 = -D \cdot [\sin^2(h)(3 + \sin(8h))] = 0 \\ D \cdot P_3 = r_3 - 2hr_4 = 0 \\ |r|^2 - 1 = 0 \end{cases}$$

By Lemma 54, $\mathrm{Ball}(P)$ *is regular at its solutions. Hence, we can use a certified numerical solver to check that the singular points of the plane projection of $\mathfrak{C}$ are characterized by the solutions of* $\mathrm{Ball}(P)$ *(Corollary 49).*

5. Checking assumptions

In this section we present Semi-algorithm 3 for checking the assumptions of Section 2. This semi-algorithm stops if and only if all the assumptions are satisfied. The main idea of the semi-algorithm comes from Lemmas 38 and 54. We use interval arithmetic as the main tool (see for example [Neu91, MKC09a]) to represent and compute with the given functions and matrices like P, $\mathrm{Ball}(P)$, J_P or $J_{\mathrm{Ball}(P)}$. In Section 5.1, we present the basics of interval arithmetic with the notation and definitions by Lin and Yap [LY11].

5.1. Interval arithmetic

Recall that for some positive integer k, by a closed (resp. open) k-box $\mathfrak{B}$, we mean the Cartesian product of k closed (resp. open) intervals. The width of a box $\mathfrak{B}$, denoted by $w(\mathfrak{B})$, is the maximal length of the intervals of that product. For a subset $A \subset \mathbb{R}^k$, the set IA is the set of all closed k-boxes that are contained in A. For the positive integer m and a function $f : A \to \mathbb{R}^m$, the function $\Box f : IA \to I\mathbb{R}^m$ is called an inclusion of f if the set $f(\mathfrak{B}) = \{f(x) \mid x \in \mathfrak{B}\}$ is contained in $\Box f(\mathfrak{B})$, for all $\mathfrak{B} \in IA$. An inclusion $\Box f$ of f is called a box function, if for any descending sequence of closed k-boxes $\mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \dots$ that converges to a point $q \in \mathbb{R}^k$, the sequence $\Box f(\mathfrak{B}_1) \supset \Box f(\mathfrak{B}_2) \supset \dots$ converges to $f(q)$. In the rest of this section, we assume that we are given a box function $\Box f$ for any function f we consider. The command *subdivide* is applied to a closed k-box $\overline{\mathfrak{B}}$, and it returns the set of boxes obtained by bisecting $\overline{\mathfrak{B}}$ in all dimensions.

An interval matrix $\Box M$ is a matrix whose coefficients are intervals. It can also be seen as the set of all matrices whose (i, j)-th coefficients belong to the (i, j)-th interval. The rank of an interval matrix $\Box M$, denoted by $\operatorname{rank}(\Box M)$, is the minimum of the ranks of all the matrices in this set.

5.2. Semi-algorithm

This section is dedicated to prove the following theorem:

Theorem 57. *For an open n-box B and a smooth function P from $\overline{B}$ to $\mathbb{R}^{n-1}$, Semi-algorithm 3 stops if and only if P satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ in $\overline{B}$.*

To check whether a given function P satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ in $\overline{B}$, we use their relation to the solutions of $\operatorname{Ball}(P)$ studied in the previous sections. Recall that for any subset $A \subseteq \mathbb{R}^n$, we defined $A_{\text{Ball}} = \{(x_1, x_2, y, r, t) \mid t \geqslant 0, (x_1, x_2, y + r\sqrt{t}), (x_1, x_2, y - r\sqrt{t}) \in A, \|r\|^2 = 1\}$. Let B be an open n-box and P be a smooth function from $\overline{B}$ to $\mathbb{R}^{n-1}$ that satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$. Consider the following assumptions:

$\aleph_1$ All solutions of $\operatorname{Ball}(P)$ in $\overline{B}_{\text{Ball}}$ are regular.

$\aleph_2$ For every solution X of $\operatorname{Ball}(P)$ in $\overline{B}_{\text{Ball}}$, none of the points of the pair $\Omega_P(X)$ (Definition 39) is in the boundary of B.

$\aleph_3$ No two distinct solutions of $\operatorname{Ball}(P)$ in $\overline{B}_{\text{Ball}}$, except the twin solutions (Remark 42), have the same plane projection.

Lemma 58. *Let B be an open n-box and P be a smooth function from $\overline{B}$ to $\mathbb{R}^{n-1}$ that satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$. Then, Assumptions $\aleph_1$, $\aleph_2$ and $\aleph_3$ are satisfied if and only if Assumptions $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are satisfied in $\overline{B}$.*

Proof. If Assumptions $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are satisfied in $\overline{B}$, then by Theorem 51 we have Assumption $\aleph_1$ is satisfied. Moreover, by Assumptions $\mathcal{A}_2$ and $\mathcal{A}_4$ we have that none of $\mathfrak{L}'_{\text{n}}$, $\mathfrak{L}'_{\text{c}}$ intersects ∂B. By Definition 39, for any solution X of $\operatorname{Ball}(P)$, we have that the points of the pair $\Omega_P(X)$ are in $\mathfrak{L}'_{\text{n}} \cup \mathfrak{L}'_{\text{c}}$ and hence are not in ∂B which implies that Assumption $\aleph_2$ is satisfied. Assume that Assumption $\aleph_3$ is not satisfied, that is, there exist two distinct non-twin solutions X, X' that have the same plane projection $p \in \mathbb{R}^2$. Let $(q_1, q_2) = \Omega_P(X)$ and $(q'_1, q'_2) = \Omega_P(X')$. By Lemma 38, the pairs $(q_1, q_2), (q'_1, q'_2)$ are distinct and the points q_1, q_2, q'_1, q'_2 have the same plane projection p. By Assumption $\mathcal{A}_3$, we cannot have three pairwise distinct points among q_1, q_2, q'_1, q'_2. Moreover, if the multiplicity at all of the points q_1, q_2, q'_1, q'_2 is one, then $(q_1, q_2), (q'_1, q'_2)$ are in $\widehat{\mathfrak{L}}_{\text{n}}$ and not distinct. Hence, at least a point say q_1 has multiplicity larger than one, i.e., $q_1 \in \mathfrak{L}_{\rm c}$ (Lemma 8). Hence, the number of solutions counted with multiplicity is at least three which contradicts Assumption $\mathcal{A}_3$. Hence, Assumption $\aleph_3$ is satisfied.

Now, assume that Assumptions $\aleph_1$, $\aleph_2$ and $\aleph_3$ are satisfied. Since, by Assumption $\aleph_1$, $\mathrm{Ball}(P)$ is a regular square system, its solution set is a zero-dimensional manifold in the compact set $\overline{B}_{\mathrm{Ball}(P)}$ (regular value theorem). Hence, $\mathrm{Ball}(P)$ has a finite number of solutions in $\overline{B}_{\mathrm{Ball}}$. Since Ω_P (Definition 39) is surjective (Lemma 40), the set $\widehat{\mathfrak{L}}$ (Definition 37) is also finite. Hence, the set $\mathfrak{L}_{\rm c} \cup \mathfrak{L}_{\rm n}$ is finite (since $\mathfrak{L}_{\rm c} \cup \mathfrak{L}_{\rm n}$ is the image of $\widehat{\mathfrak{L}}$ under the surjective function $(q_1, q_2) \to q_1$). Moreover, by Assumption $\aleph_2$, the set $\mathfrak{L}'_{\rm n} \cup \mathfrak{L}'_{\rm c}$ does not intersects the boundary of B. Hence, Assumption $\mathcal{A}_2$ and $\mathcal{A}_4$ are satisfied in $\overline{B}$. To prove that Assumption $\mathcal{A}_3$ is satisfied, let $p = (\alpha, \beta) \in \pi_{\mathfrak{C}}(\mathfrak{C})$ and $|\pi^{-1}(p)| \geqslant 3$. For pairwise distinct points $q_1, q_2, q_3 \in \pi^{-1}(p)$, by Lemma 38, we have that there exist two distinct non-twin solutions X, X' of $\mathrm{Ball}(P)$, with $\Omega_P(X) = (q_1, q_2)$ and $\Omega_P(X') = (q_1, q_3)$ such that we have the same plane projection p which contradicts Assumption $\aleph_3$. Hence, $\pi_{\mathfrak{C}}^{-1}(p)$ consists of at most two distinct points. We consider two cases:

(a) $\pi_{\mathfrak{C}}^{-1}(p)$ has two distinct elements, say q_1, q_2. By Lemma 38, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_{\rm n}$, and hence, there exists a solution $X = (\alpha, \beta, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ of $\mathrm{Ball}(P)$, with $t \neq 0$ and $\Omega_P(X) = (q_1, q_2)$. Since X is a regular solution (Assumption $\aleph_1$), by Lemma 54 we have that none of q_1, q_2 is in $\mathfrak{L}_{\rm c}$. Hence, by Lemma 8, the multiplicity of $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ at q_1 (resp. q_2) is one. Thus, the number of solutions counted with multiplicity is two.

(b) $\pi_{\mathfrak{C}}^{-1}(p)$ has a unique point q. Let m denote the multiplicity of the system $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ at q. If $m = 1$, then we are done. If $m > 1$, then by Lemma 8 we have that $q \in \mathfrak{L}_{\rm c}$. Hence, there exists a solution of $\mathrm{Ball}(P)$ of the form $X = (\alpha, \beta, y, r, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ such that $\Omega_P(X) = (q, q)$ (Lemma 40). Since X is regular (Assumption $\aleph_1$), by Lemma 54 we have that (q, q) satisfies assumption $\mathcal{A}'_5$. In particular, the multiplicity m is equal to two.

Thus, for all $p \in \pi_{\mathfrak{C}}(\mathfrak{C})$ the sum of the multiplicities of the solutions in the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ is at most two, i.e., Assumption $\mathcal{A}_3$ is satisfied. Now, Since Assumptions $\mathcal{A}_1$,$\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are satisfied and since all solutions of $\mathrm{Ball}(P)$ are regular, by Theorem 51, we have that Assumption $\mathcal{A}_5$ is also satisfied. $\square$

Using Lemma 58, we are ready to check Assumptions $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ using $\aleph_1$, $\aleph_2$ and $\aleph_3$. Since Lemma 58 requires Assumption $\mathcal{A}_1$, we start by checking that assumption with Semi-algorithm 1 that is based on subdivision.

Semi-algorithm 1 Checking Assumption $\mathcal{A}_1$

Input: An integer $n > 2$, an open n-box B and a function P from $\overline{B}$ to $\mathbb{R}^{n-1}$.

Output: True if and only if P satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$.

$L := \{\overline{B}\}$
while $L \neq \emptyset$ **do**
    $\mathfrak{B} := pop(L)$
    if $0 \in \square P(\mathfrak{B})$ and $\mathrm{rank}(\square J_P(\mathfrak{B})) < n - 1$ **then**

5: Subdivide $\mathfrak{B}$ and add its children to L.

6: **return** True.

Lemma 59. *Semi-algorithm 1 stops if and only if P satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$.*

Proof. If Semi-algorithm 1 stops, by the conditions in Step (4), the box $\overline{B}$ is partitioned into two sets of boxes. A set of boxes that are disjoint with $\overline{\mathfrak{C}}$ and the other one is a set of boxes that contain parts of $\overline{\mathfrak{C}}$ that satisfy Assumption $\mathcal{A}_1$. Thus, Assumption $\mathcal{A}_1$ is satisfied in $\overline{B}$. On the other hand, assume that P satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$ and Semi-algorithm 1 does not stop, then, for every positive real ϵ there exists a closed box $\overline{\mathfrak{B}}_\epsilon \subset \overline{B}$, with $w(\overline{\mathfrak{B}}_\epsilon) < \epsilon$ such that the conditions in Step (4) are satisfied in $\overline{\mathfrak{B}}_\epsilon$. Consider the infinite chain $\overline{\mathfrak{B}}_{\frac{1}{1}}, \overline{\mathfrak{B}}_{\frac{1}{2}}, \overline{\mathfrak{B}}_{\frac{1}{3}} \ldots$ and take $q_k \in \mathfrak{B}_{\frac{1}{k}}$, with $q_k \neq q_{k'}$ for $k \neq k'$. Since $\overline{B}$ is compact, then there exists a subsequence of q_k that converges to a point on $\overline{B}$ say q. Since $\square P$ and $\square J_P$ are box function we must have that $P(q) = 0$ and $\operatorname{rank}(J_P(q)) < n-1$. Thus, q is a point in $\overline{\mathfrak{C}}$ that does not satisfy Assumption $\mathcal{A}_1$ which is a contradiction. Hence, Semi-algorithm 1 stops. $\square$

The next step is to check Assumptions $\aleph_1$ and $\aleph_2$. For this goal, we want to find a finite set of pairwise disjoint boxes in $\overline{B}_{\mathrm{Ball}}$ such that every box contains at most one solution of $\mathrm{Ball}(P)$ and the union of these boxes contains all solutions of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$. Notice that, by the definition of box functions, for a closed $(2n-1)$-box $\overline{\mathfrak{U}}$, if $0 \notin \square\mathrm{Ball}(P)(\overline{\mathfrak{U}})$, then $\overline{\mathfrak{U}}$ does not contain a solution of $\mathrm{Ball}(P)$, whereas the condition $0 \in \square\mathrm{Ball}(P)(\overline{\mathfrak{U}})$ does not necessarily imply that a solution is in $\overline{\mathfrak{U}}$. This is why the set we are going to find might have unnecessary boxes. However, we will see later that this is enough for our purpose. Before introducing Semi-Algorithm 2, we define the following functions.

Definition 60. *Consider the set $\mathbb{R}^{2n-1}_{t\geqslant 0} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geqslant 0\}$ and define*

$$\begin{aligned} f^+_{\mathrm{Ball}} &: \mathbb{R}^{2n-1}_{t\geqslant 0} \to \mathbb{R}^n \\ (x_1, x_2, y, r, t) &\mapsto (x_1, x_2, y + r\sqrt{t}) \end{aligned}$$

and

$$\begin{aligned} f^-_{\mathrm{Ball}} &: \mathbb{R}^{2n-1}_{t\geqslant 0} \to \mathbb{R}^n \\ (x_1, x_2, y, r, t) &\mapsto (x_1, x_2, y - r\sqrt{t}). \end{aligned}$$

Define the function $f_{\mathrm{Ball}} : \mathbb{R}^{2n-1}_{t\geqslant 0} \to \mathbb{R}^n \times \mathbb{R}^n$ that maps X to $(f^+_{\mathrm{Ball}}(X), f^-_{\mathrm{Ball}}(X))$. Notice that f_{Ball} is an extension of Ω_P (Definition 39).

Semi-algorithm 2 Isolating the solutions of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$ (under Assumption $\mathcal{A}_1$)

Input: An integer $n > 2$, an open n-box B, a function P from $\overline{B}$ to $\mathbb{R}^{n-1}$ such that P satisfies Assumption $\mathcal{A}_1$ in $\overline{B}$ and a $(2n-1)$-closed box $\overline{\mathfrak{U}}_0$ that contains $\overline{B}_{\mathrm{Ball}}$ (see Remark 63).

Output: (If Semi-algorithm 2 terminates) A finite set of pairwise disjoint (see Remark 61) $(2n-1)$-boxes such that every solution of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$ is contained in a box of this set and each box contains at most one solution.

1: *Solutions* $= \emptyset$.

2: $L := \{\overline{\mathfrak{U}}_0\}$.

3: **while** $L \neq \emptyset$ **do**

4: $\quad\overline{\mathfrak{U}} := pop(L)$.

5: $\quad$**if** $0 \notin \square\mathrm{Ball}(P)(\overline{\mathfrak{U}})$ or $(\square f_{\mathrm{Ball}}(\overline{\mathfrak{U}})) \cap (\overline{B} \times \overline{B}) = \emptyset$ **then**

6: $\qquad$Continue.

7: $\quad$**if** $\mathrm{rank}(\square J_{\mathrm{Ball}(P)}(\overline{\mathfrak{U}})) = 2n-1$ and $\square f_{\mathrm{Ball}}(\overline{\mathfrak{U}}) \subset B \times B$ **then**

8: $\qquad$Add $\overline{\mathfrak{U}}$ to *Solutions*.

9: $\quad$**else**

10: $\qquad$Subdivide $\overline{\mathfrak{U}}$ and add its children to L.

11: **return** *Solutions*

Remark 61. *Notice that the output of Semi-algorithm 2, as described in the algorithmic part, may not be a set of pairwise disjoint boxes. More precisely, two boxes of the output may intersect in their boundaries if a solution of the Ball system is on or near their common boundary. To solve this issue, one could use the so-called ε-inflation (see for instance [Sta95, §5.9.1][Kea97]), but we just sketch a simple method. For every connected component of* Solutions *(i.e., an inclusion-wise maximal subset of* Solutions *such that the union of its boxes is a connected set) that contains more than one box, we compute the smallest box $\overline{\mathfrak{U}}'$ that contains this component and we remove the boxes of this connected component from* Solutions. *After shifting the grid with a small enough $\epsilon > 0$, we subdivide $\overline{\mathfrak{U}}'$ in smaller boxes and add them to* Solutions. *We then repeat Semi-algorithm 2 starting from Step (2) such that L is assigned to the modified set* Solutions. *We repeat the same process as long as we have a non-empty intersection of two boxes in* Solutions.

We assume that the process described in this remark is part of Semi-algorithm 2.

Lemma 62. *Under Assumption $\mathcal{A}_1$ in $\overline{B}$, if Semi-algorithm 2 stops, it returns a finite set* Solutions *of pairwise disjoint $(2n-1)$-boxes such that every solution of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$ is contained in a box of this set and each box contains at most one solution. Moreover, Semi-algorithm 2 stops if and only if $\mathrm{Ball}(P)$ satisfies Assumptions $\aleph_1$, $\aleph_2$ in $\overline{B}_{\mathrm{Ball}}$.*

Proof. First let us prove the correctness. Assume that Semi-algorithm 2 stops and *Solutions* is the output set. Then, by Remark 61, we have that the boxes in *Solutions* are pairwise disjoint. Moreover, $\overline{B}_{\mathrm{Ball}}$ is covered by a set of boxes such that every box of this set satisfies:

(a) one of the conditions in Step (5) which implies that no interesting solutions (i.e., that characterize singular points in $\pi_{\mathfrak{C}}(\mathfrak{C})$) are in this box, or

(b) the conditions in Step (7) which guarantee that if a solution X exists in this box, then it is regular and $\Omega_P(X) \in B \times B$. Thus, X satisfies Assumptions $\aleph_1$ and $\aleph_2$.

Hence, every solution of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$ is regular and contained in a box of *Solutions*. The condition $\mathrm{rank}(\square J_{\mathrm{Ball}(P)}(\overline{\mathfrak{U}})) = 2n-1$ guarantees that each box of *Solutions* contains at most one solution of $\mathrm{Ball}(P)$ [Sny92, Theorem A.1].

To prove the equivalence, assume that Semi-algorithm 2 stops and returns *Solutions*. According to the correctness proof, every solution X of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$ is regular and satisfies $\Omega_P(X) \in B \times B$. Thus, Assumptions $\aleph_1$ and $\aleph_2$ are satisfied in $\overline{B}_{\mathrm{Ball}}$.

On the other hand, assume that $\aleph_1$ and $\aleph_2$ hold. We prove that Semi-Algorithm 2 stops in two steps:

Step 1: By Assumption $\aleph_1$ all solutions of the square system $\mathrm{Ball}(P)$ are regular. Hence, they form a zero dimensional manifold in the compact space $\overline{B}_{\mathrm{Ball}}$. Thus, the solution set is finite. Consider $\overline{\mathfrak{U}}'$ mentioned in Remark 61 that contains the boxes of a connected component of *Solution* with more than one box. Notice that if $\overline{\mathfrak{U}}'$ is in the interior of $\overline{B}_{\mathrm{Ball}}$, then the boundary of $\overline{\mathfrak{U}}'$, shared with the boundary of the connected component, cannot have a solution of $\mathrm{Ball}(P)$. Otherwise, a neighbour box of the connected component is in *Solution* which contradicts the maximality of the connected component. Hence, after changing the grid finitely many times (that is done in the process described in Remark 61), every solution is contained in the interior of a box in *Solution*, except those solutions that are in the boundary of $\overline{B}_{\mathrm{Ball}}$. Thus, after repeating the process in Remark 61 finitely many times, the boxes of *Solutions* are pairwise disjoint.

Step 2: We prove that for any box $\overline{\mathfrak{U}} \in L$ with a small enough width, one of the conditions in Step (5) or the conditions in Steps (7) are satisfied. Thus, in both cases $\overline{\mathfrak{U}}$ will be removed from L, and hence, Semi-algorithm 2 stops after a finite number of iterations. If a small enough box of L does not contain an interesting solution of $\mathrm{Ball}(P)$, then it satisfies one of the conditions of Step (5). Let X be a solution of $\mathrm{Ball}(P)$ in $\overline{B}_{\mathrm{Ball}}$. By Assumption $\aleph_2$, we have $f_{\mathrm{Ball}}(X) \in B \times B$. Hence, by the continuity of f_{Ball} there must exist a box $\overline{\mathfrak{U}}' \in L$ (in some iteration of the while-loop) with a small enough width that contains X and satisfies $\square f_{\mathrm{Ball}}(\overline{\mathfrak{U}}') \subseteq B \times B$. By Assumption $\aleph_1$, X is a regular solution. Assume that for any box $\overline{\mathfrak{U}} \in L$ that contains X the rank of $\square J_{\mathrm{Ball}(P)}(\overline{\mathfrak{U}})$ is less than $2n-1$. Let $\mathfrak{U}_1 \supset \mathfrak{U}_2 \ldots$ be a chain of those boxes. Since $\square J_{\mathrm{Ball}(P)}$ is a box function, we have that $\lim_{i\to\infty} \square J_{\mathrm{Ball}(P)}(\mathfrak{U}_i) = J_{\mathrm{Ball}(P)}(X)$. Consider the box function $\det(\square J_{\mathrm{Ball}(P)})$. Notice that $\det(\square J_{\mathrm{Ball}(P)}(\mathfrak{U}_i))$ converges to $\det(J_{\mathrm{Ball}(P)}(X))$. However, $0 \in \det(\square J_{\mathrm{Ball}(P)}(\mathfrak{U}_i))$ for all $i \in \mathbb{N}^*$ but $\det(J_{\mathrm{Ball}(P)}(X)) \neq 0$ (Assumption $\aleph_1$) which contradicts the fact that $\det(\square J_{\mathrm{Ball}(P)}(\mathfrak{U}_i))$ is a box function of $\det(J_{\mathrm{Ball}(P)})$. Thus there exists a box that contains X such that the rank of $\square J_{\mathrm{Ball}(P)}$ is $2n-1$.

Thus, For any box in L with a small enough width one of the conditions of Step (5) is satisfied or all of the conditions in Step (7) are satisfied which proves the lemma. Hence, Semi-algorithm 2 terminates. $\square$

Remark 63. *Semi-algorithm 2 requires a closed $(2n-1)$-box $\overline{\mathfrak{U}}_0$ that contains $\overline{B}_{\mathrm{Ball}}$. For instance the following set could be used: $\{(q,r,t) \in \mathbb{R}^{2n-1} \mid q \in \overline{B}, -1 \leqslant r_i \leqslant 1$ for $i \in \{3,\ldots,n\}, 0 \leqslant t \leqslant \frac{\xi^2}{4}\}$ with $\xi = \max\{\|q-q'\| \mid q, q' \in \overline{B}\}$.*

Finally, using Lemma 58, Semi-algorithm 3 checks whether P satisfies Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ in $\overline{B}$.

Semi-algorithm 3 Checking Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$

Input: An open n-box B and a smooth function P from $\overline{B}$ to $\mathbb{R}^{n-1}$.

Output: True, if and only if P satisfies Assumption $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ in $\overline{B}$.

1: Check Assumption $\mathcal{A}_1$ (Semi-algorithm 1).
2: Compute a closed $(2n-1)$-box $\overline{\mathfrak{U}}_0$ that contains $\overline{B}_{\text{Ball}}$ (Remark 63).
3: $L :=$ the output of Semi-algorithm 2.
4: **for** all distinct $\overline{\mathfrak{U}}, \overline{\mathfrak{U}}' \in L$ **do**
5: Keep refining $\overline{\mathfrak{U}}, \overline{\mathfrak{U}}'$ until their plane projections are disjoint (ignoring the twin solutions) or it is guaranteed that one of them has no solution of $\text{Ball}(P)$.
6: **return** True.

Remark 64. *In Step (5), by refining we mean that we subdivide both $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{U}}'$ until one of the mentioned conditions in Step (5) is satisfied. Analogously to the process described in Remark 61, we can subdivide in such a way that if a solution of $\text{Ball}(P)$ in $\overline{\mathfrak{U}}$ (resp. $\overline{\mathfrak{U}}'$) exists, then it is contained in a unique child of $\overline{\mathfrak{U}}$ (resp. $\overline{\mathfrak{U}}'$).*

5.3. Isolation of singularities

If Semi-algorithm 3 stops, L is a set of pairwise disjoint boxes each of which containing at most one solution of $\text{Ball}(P)$. If a box of L does not intersect the hyperplane $t = 0$ on its boundary, then an interval existence test such as Miranda test [X] or the interval Newton operator [Neu91, Section 5.1] can be applied together with further refinements to conclude whether the box actually contains a solution of $\text{Ball}(P)$ or not. When the box isolates a solution, this solution then projects to a node of the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$. On the other hand, if a box of L intersects the hyperplane $t = 0$ on its boundary, since an existence test cannot decide the existence of a solution on the boundary of a box, it cannot decide whether the box contains a solution of $\text{Ball}(P)$ or not.

One may wish to solve independently the Ball system with the additional constraint $t = 0$ to identify cusps. Unfortunately, in this case $\text{Ball}(P)$ is an overdetermined system and thus it is difficult to certify its solutions numerically. However, in the special case of a silhouette curve defined by P (using the notation in Definition 26), if $X \in \overline{B}_{\text{Ball}}$, with $t = 0$, the equations $D \cdot P_i(X) = 0$ for $1 \leqslant i \leqslant n-2$ imply the equation $P_{n-1}(X) = 0$. Moreover, the functions P_{n-1} and $S \cdot P_{n-1}$ coincide for $t = 0$. Hence, in the set $\{X \in \overline{B}_{\text{Ball}} \mid t = 0\}$, the system $\text{Ball}(P)$ without the equation $P_{n-1}(X) = 0$ is a square system of $2n-2$ equations. In the case where $n = 3$, it is proved in [IMP16a, Lemmas 9 & 10] that this system is regular iff the solution projects to an ordinary cusp. However, for $n > 3$, the regularity of $\text{Ball}(P)$ is not clear and we leave it as a conjecture.

6. Conclusion

We propose a regular square system that encodes the singularities of the plane projection of a generic curve in $\mathbb{R}^n$. The genericity of the assumptions is proved via transversality theory. For the silhouette case, the genericity of a part of the assumptions is proved and we state the missing part as Conjectures 32 & 33. We provide a semi-algorithm that checks whether a given system satisfies the generic assumptions. The cost of our approach is that the number of variables is doubled, this is a drawback for subdivision methods that are exponential in the dimension. One way to overcome this issue is to restrict the search domain for the Ball system. Similarly as in the 3-dimensional case

[IMP18], the smooth curve in $\mathbb{R}^n$ could first be enclosed in a set of boxes by a certified tracking. For the computation of nodes, the Ball system could then only be solved in the small $(2n-1)$-dimensional domains corresponding to enclosing boxes of the curve that overlap in projection.

References

[AGZV12] Vladimir Igorevich Arnold, Sabir Medgidovich Gusein-Zade, and Alexander Nikolaevich Varchenko, *Singularities of differentiable maps. Volume 1*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012. MR2896292

[Bôc64] Maxime Bôcher, *Introduction to higher algebra*, Dover Publications, Inc., New York, 1964.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy, *Real algebraic geometry*, Springer, 1998.

[BL13] Carlos Beltrán and Anton Leykin, *Robust certified numerical homotopy tracking*, Foundations of Computational Mathematics **13** (2013), no. 2, 253–295.

[BLM+16] Yacine Bouzidi, Sylvain Lazard, Guillaume Moroz, Marc Pouget, Fabrice Rouillier, and Michael Sagraloff, *Solving bivariate systems using rational univariate representations*, Journal of Complexity **37** (2016), 34 –75.

[BPR06] Saugata Basu, Pollack Pollack, and Marie-Françoise Roy, *Algorithms in real algebraic geometry*, 2nd ed., Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, 2006.

[BV88] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988.

[CLO05] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, Second, Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005. MR2122859

[CLO92] ———, *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*, Undergraduate texts in mathematics, Springer-Verlag New York-Berlin-Paris, 1992.

[Dem00] Michel Demazure, *Bifurcations and catastrophes*, Universitext, Springer-Verlag, Berlin, 2000. Geometry of solutions to nonlinear problems, Translated from the 1989 French original by David Chillingworth.

[DL14] Nicolas Delanoue and Sébastien Lagrange, *A numerical approach to compute the topology of the apparent contour of a smooth mapping from R^2 to R^2*, Journal of Computational and Applied Mathematics **271** (2014), 267 –284.

[DLZ11] Barry Dayton, Tien-Yien Li, and Zhonggang Zeng, *Multiple zeros of nonlinear systems*, Math. Comp. **80** (2011), no. 276, 2143–2168.

[GG73] Marty Golubitsky and Victor Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York-Heidelberg, 1973. Graduate Texts in Mathematics, Vol. 14.

[GVK96] Laureano González-Vega and M'hammed El Kahoui, *An improved upper complexity bound for the topology computation of a real algebraic plane curve*, Journal of Complexity **12** (1996), no. 4, 527 –544.

[Hon96] Hoon Hong, *An efficient method for analyzing the topology of plane real algebraic curves*, Mathematics and Computers in Simulation **42** (1996), no. 4, 571 –582. Sybolic Computation, New Trends and Developments.

[IMP16a] Rémi Imbach, Guillaume Moroz, and Marc Pouget, *A certified numerical algorithm for the topology of resultant and discriminant curves*, Journal of Symbolic Computation **80, Part 2** (2016), 285–306.

[IMP16b] Rémi Imbach, Guillaume Moroz, and Marc Pouget, *Numeric and certified isolation of the singularities of the projection of a smooth space curve*, Mathematical aspects of computer and information sciences, 2016, pp. 78–92.

[IMP18] Rémi Imbach, Guillaume Moroz, and Marc Pouget, *Reliable location with respect to the projection of a smooth space curve*, Reliab. Comput. **26** (2018), 13–55. MR3804276

[Kea97] Ralph Baker Kearfott, *Empirical evaluation of innovations in interval branch and bound algorithms for nonlinear systems*, SIAM Journal on Scientific Computing **18** (1997), no. 2, 574–594.

[LY11] Long Lin and Chee Yap, *Adaptive isotopic approximation of nonsingular curves: the parameterizability and nonlocal isotopy approach*, Discrete & Computational Geometry **45** (2011Jun), no. 4, 760–795.

[MKC09a] Ramon Moore, Baker Kearfott, and Michael Cloud, *Introduction to interval analysis*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.

[MKC09b] Ramon E Moore, R Baker Kearfott, and Michael J Cloud, *Introduction to interval analysis*, Siam, 2009.

[Neu91] Arnold Neumaier, *Interval methods for systems of equations*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1991.

[Sny92] John M. Snyder, *Interval analysis for computer graphics*, SIGGRAPH Comput. Graph. **26** (July 1992), no. 2, 121–130.

[Sta95] Volker Stahl, *Interval methods for bounding the range of polynomials and solving systems of nonlinear equations*, Ph.D. Thesis, 1995.

[vdHL18] Joris van der Hoeven and Robin Larrieu, *Fast reduction of bivariate polynomials with respect to sufficiently regular gröbner bases*, Proceedings of the 2018 acm international symposium on symbolic and algebraic computation, 2018, pp. 199–206.

[Whi43] Hassler Whitney, *Differentiable even functions*, Duke Mathematical Journal **10** (1943), 159–160.

[Whi65] ______, *Local properties of analytic varieties*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), 1965, pp. 205–244. MR0188486

[XY19] Juan Xu and Chee Yap, *Effective subdivision algorithm for isolating zeros of real systems of equations, with complexity analysis*, Proceedings of the 2019 international symposium on symbolic and algebraic computation, 2019, pp. 355–362.